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ON MAJORANTS FOR SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS IN REGIONS OF THE COMPLEX PLANE

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In this paper, we investigate the rate of growth of functions which are analytic in an arbitrary simplyconnected region of the complex plane and which are solutions of first order algebraic differential equations (i.e., equations of the form $\Omega(z, y, dy/dz) = 0$, where Ω is a polynomial in z, y and dy/dz).

In the course of constructing an example for second order equations, Vijayaraghavan in [5] showed that for *any* realvalued increasing function $\Phi(x)$ on the interval $(0, +\infty)$, it is possible to find a complex function $h(z)$, which is analytic in a simply-connected region R of the plane containing $(0, +\infty)$, and satisfies a first order algebraic differential equation, and which has the property that $|h(x)| > \Phi(x)$ at a sequence of real x tending to $+\infty$[1]. For a given $\Phi(x)$, the function $h(z)$ constructed was of the form $P(az)$ where $P(u)$ is the Weierstrass P-function with primitive periods w and iw' (w, w' real), and where the constant a was of the form $a = w + ib$, where b depends on Φ and b/w' is irrational. Since $P(az)$ has poles at all points $(mw/a) + (niw'/a)$ where m and n are integers, clearly the region R associated with the solution $h(z) = P(az)$ depends on a and hence on $\Phi(x)$. A natural question is thus raised, namely, can such examples be constructed where, for all $\Phi(x)$, the simply-connected region R remains the same. That is, does there exist a simply-connected region R containing $(0, +\infty)$ with the property that for any real-valued increasing function $\Phi(x)$ on $(0, +\infty)$, there is a solution $h(z)$, analytic on R, of a first order algebraic differential equation, such that $|h(x)| > \Phi(x)$ at a sequence of real x tending to $+\infty$? In this paper we answer this question in the *negative* by proving the following more general result (§2 below): If R is any simply-connected region, then there exists a real-valued continuous function $\Psi(z)$ on R with the property that for any function $h(z)$, analytic on R, which satisfies a first order algebraic differential equation, there is a compact set K contained in R such that $|h(z)| < \Psi(z)$ on $R-K$. In the case where R is not the whole plane, we show that $\Psi(z)$ may be taken to be

[1] None of the solutions $h(z)$ constructed by Vijayaraghavan are real-valued on any interval $(x_0, +\infty)$. Of course this is in accord with the well-known result of Lindelöf [**2**; p. 213] that a *real-valued* solution on an interval $(x_0, +\infty)$ is majorized, on some interval $(x_1, +\infty)$, by $\exp(\exp x)$.