PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 2, 1970

PROJECTIVE DISTRIBUTIVE LATTICES

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It will be shown that a countable distributive lattice is projective if and only if the product of any two join irreducible elements is join irreducible, and every element of the lattice is both a finite sum of join irreducible elements and a finite product of meet irreducible elements. For an arbitrary distributive lattice, necessary and sufficient conditions for projectivity are obtained by adding to these conditions a further condition on the set of join irreducible elements.

1. Definitions. We use sum and product notation for least upper bounds and greatest lower bounds. If A and B are meet semilattices, then a meet homomorphism $f: A \rightarrow B$ is a function such that $f(xy) = f(x)f(y)$. An element x of a lattice is called join irreducible if $x = y + z$ implies $x = y$ or $x = z$. x is called sub join irreducible if $x \leqq y + z$ implies $x \leqq y$ or $x \leqq z$. In a distributive lattice these two notions coincide. We define meet irreducible and super meet irreducible in a dual manner. A lattice is called conditionally implicative if whenever $x \not\leqq y$ there exists a largest z such that $xz \leqq y$. The smallest and largest element of a lattice are denoted by 0 and 1 respectively. The cardinal of a set S is denoted by $|S|$. For definitions of projective distributive lattice and retract, see [1]. Note that epimorphisms are understood to be homomorphisms which are onto. If the term epimorphism is used as in the general theory of categories, there are no projective distributive lattices.

2. Projective distributive lattices. Consider the following properties of a lattice.

(P1) Every element is a sum of finitely many sub join irreducible elements.

(P2) Every element is a product of finitely many super meet irreducible elements.

(P3) The product of any two sub join irreducible elements is sub join irreducible.

(P4) The sum of any two super meet irreducible elements is super meet irreducible.

(P5) The lattice is conditionally implicative.

(P6) The lattice is dually conditionally implicative.

THEOREM 1. *Suppose A and B are lattices and A is a retract of B. Then we have:*