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VON NEUMANN ALGEBRAS GENERATED BY OPERATORS SIMILAR TO NORMAL OPERATORS

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A normal operator generates an abelian von Neumann algebra. However, an operator which is similar to a normal operator may generate a von Neumann algebra which is not even type I. In fact, it is shown that if $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space and $\mathscr{A}$ has no type II finite summand, then $\mathscr{A}$ has a generator which is similar to a self-adjoint and $\mathscr{A}$ has a generator which is similar to a unitary. The restriction that $\mathscr{A}$ have no type II finite summand can be removed provided that it is assumed that every type II finite von Neumann algebra has a single generator.

Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{A}$ be a von Neumann algebra on $\mathscr{H}$. $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$. For $n \geqq 2$, let $M_n(\mathscr{A})$ denote the von Neumann algebra of $n \times n$ matrices with entries in $\mathscr{A}$. If T is a bounded operator, the $\mathscr{R}(T)$ is the von Neumann algebra generated by T.

We begin with some lemmas.

LEMMA 1. *Let $\mathscr{A} = \mathscr{R}(C)$ and suppose $n \geqq 3$. Let $\{\lambda_k\}_{k=1}^{n}$ and $\{a_k\}_{k=1}^{n-1}$ be sequences of complex numbers such that the λ_k are distinct, each $a_k \neq 0$, and $||(\lambda_1 - \lambda_2)C|| \leqq |a_1 a_2|$. Define $A = (A_{i,j})_{i,j=1}^{n} \in M_n(\mathscr{A})$ by $A_{k,k} = \lambda_k I$, $A_{k+1,k} = a_k I$, $A_{3,1} = C$,* and *$A_{i,j} = 0$ otherwise. Define $B = (B_{i,j})_{i,j=1}^{n} \in M_n(\mathscr{A})$ by $B_{k,k} = \lambda_k I$ and $B_{i,j} = 0$ if $i \neq j$. Then A and B are similar, and $\mathscr{R}(A) = M_n(\mathscr{A})$.*

Proof. It follows from [11, Lemma 1] that $\mathscr{R}(A) = M_n(\mathscr{A})$. To show that A and B are similar we need only that the λ_k are distinct. We must find an invertible operator S such that $AS = SB$. Such an S of the form $S = I + N$, where N is lower triangular and nilpotent, can be computed easily. Merely perform the matrix multiplications and solve for the entries of S. We omit the details.

REMARK 1. If the operator $S = I + N$ in Lemma 1 is computed, we see that we can make the entries of N small by choosing $||C||$, $|a_1|, |a_2|, \cdots, |a_{n-1}|$ suitably small. Hence we can suppose that $||N|| < 1/2$. Then $||S|| = ||I + N|| < 3/2$ and $||S^{-1}|| = ||I - N + N^2 - \cdots \pm N^{n-1}|| < 2$. Note also that by choosing $||C||, |a_1|, |a_2|, \cdots, |a_{n-1}|$ suitably, we can assume that $||A|| \leqq ||B|| + 1$.