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WHICH LINEAR MAPS OF THE DISK ALGEBRA ARE MULTIPLICATIVE?

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Let T be a linear map of the disk algebra into itself which is of norm one and fixes the constants. This paper considers the question of what additional restrictions suffice to insure that T is multiplicative. It is shown that if T is an isometry and the range of T is a ring then T is multiplicative and that if the image under T of the coordinate function of the disk is an extreme point of the unit ball of the disk algebra then T is multiplicative.

Let D be the closed unit disk of the complex plane and A the disk algebra, the supremum normed Banach algebra of functions continuous on D and analytic in the interior of D. Let Z be the identity function on D, $Z(x) = x$ for all x in D. Let L be the set of extreme points of the closed unit ball of A. Let K be the set of linear maps of A into itself which are of norm less than or equal one and which fix the constants. Let K' be the set of elements of K which are multiplicative; $K' = \{T;\ T \text{ in } K,\ T(fg) = T(f)T(g) \text{ for all } f \text{ and } g \text{ in } A\}$. We note in passing that any continuous non-zero multiplicative linear map of A into itself is of norm one and hence in K' and that any such map is given by composition with an element of A; that is if T is K' then $Tf = f \circ T(Z)$ for all f in A. The first of these facts is a general function algebra result and the second follows from the fact that the polynomials are dense in A.

We will investigate the question of what additional restrictions are needed to insure that T, an element of K, is actually an element of K'. We will prove

THEOREM A. *If T in K is an isometry* (i.e., $\|Tf\| = \|f\|$ for all A in f) *and $T(A)$ is a ring, then T is in K'. and*

THEOREM B. *If T is in K and $T(Z)$ is in L, then T is in K'.*

Theorem A extends the known result that any surjective isometry in K is multiplicative and provides a partial answer to the question raised by Phelps ([1], pg. 354) of describing the non-surjective isometries of the disk algebra.

Phelps ([3]) has shown that if T is in K' and $T(Z)$ is in L, then T is an extreme point of K. This result combined with Theorem B gives.