PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 1, 1971

A GENERALIZATION OF COMMUTATIVE AND ASSOCIATIVE RINGS

ERWIN KLEINFELD

Let R be a ring satisfying the following three defining relations: (i) $(x, y^2, x) = y\circ(x, y, x)$, (ii) $(x, y, z) + (y, z, x) + (z, x, y) = 0$, and (iii) $((x, y), x, x) = 0$, where $(a, b, c) = (ab)c - a(bc)$, $(a, b) = ab - ba$, and $a\circ b = ab + ba$. All three identities follow from commutativity, hence are true in Jordan rings. Besides (i) holds in Lie and alternative rings, (ii) holds in Lie and quasiassociative rings and in alternative rings of characteristic three, while (iii) holds in right alternative rings. The main result is that if R has characteristic $\neq 2, 3$ (that means no elements in R have additive order two or three) and no divisors of zero then R must be either associative or commutative.

The classification of commutative rings has not been attempted, perhaps because the important tool of decomposition relative to an idempotent due to Albert requires power-associativity. It is well known that commutative rings need not be fourth power associative. Besides one would have to find a way to classify the numerous finite, commutative devision rings. The choice of identities was dictated by the fact that there exist rings without divisors of zero which satisfy (i) and (ii) but which are not commutative. The Cayley-Dickson division algebras satisfy (i) and (iii) yet are not commutative. In fact the Cayley-Dickson division algebras over fields of characteristic 3 satisfy all three identities. There is some reason to believe that ultimately there will be found a set of identities suitable for generalizing the better known rings such as alternative, Jordan and Lie rings. The present study is helpfull in delineating possible candidates for replacing commutativity. A few examples will be discussed at the end of the paper.

Throughout most of the paper we shall require R to be a ring which satisfies (i)-(iii), has characteristic $\neq 2$, 3 and no divisors of zero. However in the beginning we can dispense with (iii) and weaken the assumption of no divisors of zero to assuming that there exists no element $x \neq 0$, such that $x^2 = 0$. In every ring we have the Teichmüller identity

$$f(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0 \, . \tag{1}$$

Hence