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NOTE ON THE OPEN MAPPING THEOREM

LAWRENCE G. BROWN

The open mapping and closed graph theorems are usually stated in terms of metrizable topological groups which are complete in a one-sided uniformity. It would be desirable to use only the weaker hypothesis of completeness in the two-sided uniformity. Important results of this sort are already known, and it is our purpose to strengthen those results by removing the separability hypotheses.

There are three uniformities normally used in connection with a topological group: the left, right, and two-sided uniformities. Completeness in the left uniformity is equivalent to completeness in the right uniformity, and stronger than completeness in the two-sided uniformity. If the group is metrizable, then it always has at least one left invariant metric; completeness in the left uniformity is equivalent to completeness in this metric (it does not matter which left-invariant metric is chosen). Still in the case of a metrizable group, completeness in the two-sided uniformity is equivalent to topological completeness, i.e., the existence of at least one metric in which the group is complete as a metric space (see [3], Exercise $Q(d)$, p. 212 for proof). Every topological group has a completion in the two-sided sense, but not necessarily in the one-sided sense (more precisely, the completion in the left-uniformity, though it is a complete uniform space, is not in general a group). When the one-sided completion does exist, it coincides with the two-sided completion. Two important examples of metrizable topological groups which are complete in the two-sided but not the one-sided sense are the full permutation group of a countably infinite set and the unitary group of an infinite-dimensional, separable Hilbert space (with the strong operator topology). Both of these examples are separable. Nonseparable examples of less naturality can be obtained by taking the direct product of one of the above with a nonseparable Banach space.

Lemma 1 below, which is the separable case of our theorem, was essentially proved by Banach [1], though his statement of it (Satz 8) was weaker. Its present statement and brief proof are the same as in Corollary 3.2 of Pettis [5]. Corollary 1 of the theorem and the corollary to Lemma 1 have some independent interest. They state that if N is a closed normal subgroup of the group G, which is metrizable and complete in its two-sided uniformity, then G/N is also complete. The corresponding result for one-sided completeness is trivial and is true even if N is not normal. (This case where N is