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A FORM OF THE MOMENT PROBLEM FOR LIE GROUPS

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A form of the Hamburger moment problem for the real line is generalized and solved for an arbitrary Lie group. The solution relates the unitary representations of the Lie group to certain symmetric representations of the associated universal enveloping algebra.

The celebrated Hamburger moment problem for the real line asks: Which linear functionals on the algebra of polynomials can be realized by integrating these polynomials against a fixed positive measure? Hamburger's classic solution shows that a linear functional has this property if and only if it assigns positive values to positive polynomials [6, 10].

By taking Fourier transforms, we may rephrase the problem as follows: Which linear functionals on the algebra of distributions supported at the origin can be realized by evaluating these distributions at a fixed function of positive type? The solution shows that a linear functional has this property if and only if it assigns positive values to distributions of positive type.

In this latter form we may extend the problem to an arbitrary Lie group, and search for a solution in the same form. Here we confirm that this extended problem does indeed admit such a solution (Theorem 3). We then reformulate the result in terms of the space of unitary representations of the group to give a less satisfactory extension of the problem in its original form (Theorem 4). In the process we find that we can characterize the symmetric representations of the associated Lie algebra which can be "lifted" to unitary representations of the group (Theorem 5). As in the classical case [1], the heart of the matter lies in showing that positive linear functionals on the moment algebra are necessarily continuous in an appropriate sense (Corollary 2).

The algebra of distributions of compact support. In order to formulate the problem on a given Lie group, we shall need an algebraic structure containing both the Lie group and its Lie algebra. For this purpose we choose the algebra of all distributions with compact support defined on the group. The properties of this algebra which we shall use are all obtained by direct extensions of arguments developed by Laurent Schwartz in his investigations of distribution defined on R^n [9, also 2, 3, 4] and are only summarized here.