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CONVOLUTIONS OF ARITHMETIC FUNCTIONS OVER COHESIVE BASIC SEQUENCES

A. A. GIOIA AND D. L. GOLDSMITH

Necessary and sufficient conditions are given for the generalized convolution of two arithmetic functions which are units with respect to a basic sequence $\mathscr{B}$ to be again a unit on $\mathscr{B}$. These conditions are then investigated from function-theoretic and combinatorial points of view.

1. Introduction. The relation between the multiplicative properties of an arithmetic function and the underlying basic sequence has already been studied in some detail (see [1]). We will confine ourselves here to the consideration of a certain class of basic sequences, and an investigation of the properties of convolutions of arithmetic functions over these basic sequences. The definitions and preliminary results used here may be found in [1], but we will repeat them as a matter of convenience.

A *basic sequence* $\mathscr{B}$ is a set of pairs (a, b) of positive integers with the properties

(i) $(1, k) \in \mathscr{B}$ $(k = 1, 2, \cdots)$,
(ii) $(a, b) \in \mathscr{B}$ if and only if $(b, a) \in \mathscr{B}$,
(iii) $(a, bc) \in \mathscr{B}$ if and only if $(a, b) \in \mathscr{B}$ and $(a, c) \in \mathscr{B}$.

Some examples of basic sequences are

$$\mathscr{S} = \bigcup_{k=1}^{\infty} S_k, \text{ where } S_k = \{(1, k), (k, 1)\},$$
$$\mathscr{M} = \{(a, b) \mid a \text{ and } b \text{ are relatively prime positive integers}\},$$
$$\mathscr{L} = \{(a, b) \mid a \text{ and } b \text{ are any positive integers}\},$$
$$\mathscr{T}_p = \mathscr{S} \cup \{(p^a, p^b) \mid a \text{ and } b \text{ are any positive integers}\},$$

where p is a prime.

A pair (a, b) of positive integers is called a *primitive pair* if both a and b are primes. If $a \neq b$, the pair is a *type I* primitive pair; if $a = b$, the pair is a *type II* primitive pair. We see that a basic sequence is completely determined by the primitive pairs it contains. If Φ is any set of pairs (primitive or not) of positive integers, the basic sequence *generated* by Φ is defined to be

$$\Gamma[\Phi] = \bigcap \mathscr{D},$$

where the intersection is taken over all basic sequences $\mathscr{D}$ which contain Φ. Thus, a basic sequence is generated by its subset of