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A CONSTRUCTIVE STUDY OF MEASURE THEORY

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In this paper we study measures on locally compact metric spaces. The constructive theory of a nonnegative measure has been treated in Bishop's book "Foundations of Constructive Analysis". Unfortunately, there is no constructive method to decompose a general signed measure into a difference of two nonnegative ones. In analogy to the classical development, we shall consider two ways to look at a signed measure, namely, as a function function (an integral) and as a set function (a set measure). From an integral on a locally compact metric space X we obtain compact subsets of X to which measures can be assigned. The set measure thus arrived at is shown to be in a weak sense additive, continuous, and of bounded variation. Next we study a set measure having these three properties defined on a large class of compact subsets of X. From such a set measure we derive a linear function on the space of test functions of X. This linear function is then shown to be an integral. Finally it is demonstrated that the set measure arising from an integral gives rise in this manner to an integral which is equal to the original one. In particular, every integral is the integral arising from some measure (Riesz Representation Theorem).

We shall make use of concepts and results in Bishop's book (reformed to hereafter as C.A.), in which one can find a presentation of the constructive viewpoint and the constructive methods.

1. Compact subsets of a boundedly compact metric space. A metric space (X, d) is said to be totally bounded if, given any $\varepsilon > 0$, there is a finite, possibly empty, sequence of points in X which forms an ε-net for X. A metric space (X, d) is said to be compact if it is totally bounded and complete. A boundedly compact space[1] is one in which every bounded subset is contained in some compact subset. Hereafter let (X, d) denote such a space. For each compact subset A of X, and each $x \in X$, we let $d(x, A)$ stand for the number min (1, inf $\{d(x, y): y \in A\})$. Here the infimum is easily proved to exist (C.A.) if A is nonempty, and is taken to be ∞ if A is empty. Given any compact subsets A and B of X, write

$$d'(A, B) \equiv \max (0, \sup\{d(x, B): x \in A\}, \sup \{d(x, A): x, B\}) \text{ .}$$

Here the supremum of an empty set is taken to be $-\infty$. d' can easily be shown to be a metric on the family of compact subsets of X.

[1] Called locally compact space in C. A.