PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 2, 1972

A THEOREM ON BOUNDED ANALYTIC FUNCTIONS

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The purpose of this paper is to prove the following

THEOREM: Let $\phi_1, \phi_2, \cdots$ be an infinite sequence of functions in $L^1([0, 2\pi])$ such that $L(f) = \lim_{n\to\infty} \int_0^{2\pi} f(e^{i\theta})\phi_n(\theta)d\theta$ exists for every $f \in H^\infty$. Then there is a $\phi \in L^1([0, 2\pi])$ such that $L(f) = \int_0^{2\pi} f(e^{i\theta})\phi(\theta)d\theta$ for all $f \in H^\infty$.

Throughout this paper we will use the following notation and conventions: D will denote the unit disc and T its boundary. In order to save time we will avoid making distinctions between T and $[0, 2\pi]$ if no confusion results. Similarly, it will be convenient to treat elements of $H^\infty[= H^\infty(D)$, the bounded analytic functions on $D]$ as though they were the same as those functions on T with which they are naturally identified.

If $w \in D$, the symbol g_w will stand for the function $z \to g(wz)$. $C(T)$ will stand for the usual space of continuous functions on T. A will denote the subspace of $C(T)$ of functions analytically extendable to D. λ will denote ordinary Lebesgue measure divided by 2π and "WLOG" means "without loss of generality".

In their paper [4] Piranian, Shields, and Wells observed that the theorem stated above would imply their result, namely that if $a_0, a_1, \cdots$ was a sequence of complex constants such that $\lim_{r\to 1} \sum_{n=0}^\infty a_n b_n r^n$ exists for all $f \in H^\infty$ [with Taylor coefficients $b_0, b_1, \cdots$], then the a_n's are the the nonnegative Fourier coefficients of an $L^1([0, 2\pi])$ function. They also mentioned that our result here was a question raised in [1].

Kahane [3], using a somewhat different method than that in [4] showed that under the hypothesis of our main theorem, there was a $\phi \in L^1([0, 2\pi])$ such that the conclusion held for all $f \in A$. He went further to show that the subset of H^∞ for which the conclusion held was large in some sense. Our proof here makes use of Kahane's result.

2. Remarks and lemmas.

First, given the hypothesis of the main theorem we may assume WLOG that the ϕ_n's are uniformly bounded in L^1 norm. To see why this is so we observe that for each n, $g \longrightarrow L_n(g) = \int_T g\phi_n$ is a bounded linear functional on A. By the uniform boundedness principle, the norms of the L_n's as elements of A^* are uniformly bounded, say by M. By the Hahn-Banach Theorem, each L_n may be extended to an element of $C(T)^*$with norm less than