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SOCLE CONDITIONS FOR QF-1 RINGS

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Let R be an associative ring with 1. If ${}_RM$ is a left R-module, then M can be considered as a right $\mathscr{C}$-module, where $\mathscr{C} = \operatorname{Hom}({}_RM, {}_RM)$ is the centralizer of ${}_RM$. There is a canonic ring homomorphism ρ from R into the double centralizer $\mathscr{D} = \operatorname{Hom}(M_{\mathscr{C}}, M_{\mathscr{C}})$ of ${}_RM$. For a faithful module ${}_RM$, the homomorphism ρ is injective, and ${}_RM$ is called balanced (or to satisfy the double centralizer condition) if ρ is surjective. An artinian ring R is called a QF-1 ring if every finitely generated faithful R-module is balanced. This definition was introduced by R. M. Thrall as a generalization of quasi-Frobenius rings, and he asked for an internal characterization of QF-1 rings.

The paper establishes three properties of QF-1 rings which involve the left socle and the right socle of the ring; in particular, it is shown that QF-1 rings are very similar to QF-3 rings. The socle conditions are necessary and sufficient for a (finite dimensional) algebra with radical square zero to be QF-1, and thus give an internal characterization of such QF-1 algebras. Also, as a consequence of the socle conditions, D. R. Floyd's conjecture concerning the number of indecomposable finitely generated faithful modules over a QF-1 algebra is verified. In fact, a QF-1 algebra has at most one indecomposable finitely generated faithful module, and, in this case, is a quasi-Frobenius algebra.

An artinian ring R is called a QF-1 ring if every finitely generated faithful R-module is balanced. This definition goes back to R. M. Thrall [15] who asked for an internal description of QF-1 algebras. The aim of this paper is to prove the following theorem.

THEOREM. *Let R be a QF-1 ring with left socle L and right socle J. If e and f are primitive idempotents with $f(L \cap J)e \neq 0$, then*

(1) *either $\partial_l Je = 1$ or $\partial_r fL = 1$,*
(2) *we have $\partial_l Le \times \partial_r fJ \leqq 2$, and*
(3) *$\partial_l Le = 2$ implies $Je \subseteqq Le$.*

Here, $\partial_l I$ denotes the length of (a composition series of) the left ideal I, whereas $\partial_r K$ denotes the length of the right ideal K.

The second socle condition shows that QF-1 rings are very similar to QF-3 rings, because an artinian ring is a QF-3 ring if and only if for every pair e, f of primitive idempotents with $f(L \cap J)e \neq 0$,