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DIMENSION THEORY IN ZERO-SET SPACES

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The main purpose of this paper is to show that the zero-set spaces of Gordon provide a natural and very general setting in which to develop dimension theory. Defining covering dimension for zero-set spaces in the natural way, it is shown that the subspace theorem, the product theorem, and sum theorem hold. As a consequence it is possible to give a subspace theorem for arbitrary topological spaces.

1. A subspace theorem for arbitrary topological spaces. From the general theory in zero-set spaces it is possible to deduce a subspace theorem in arbitrary topological spaces. To express the result it is convenient to have on hand a definition of the dimension of a ring as defined in [1]. This notion also allows the simplification of certain proofs in dimension theory.

Let R be a commutative ring with identity. By a *basis* of R we mean a finite set of elements which generate R. The *order* of a basis is the largest integer n for which there exist $n+1$ members of the basis with nonzero product. A basis $\{a_i\}$ of R is said to *refine* the basis $\{b_j\}$ of R if each a_i is a multiple of some b_j. The *dimension* of R, denoted by $d(R)$, is the least cardinal $\mathfrak{m}$ such that every basis of R has a refinement of order at most $\mathfrak{m}$.

Let $C(X)$ denote the ring of continuous real-valued functions on a topological space X. It is shown in [1] that $\dim X = d(C(X))$. For subspaces A of X, the statement $d(C(A)) \leqq d(C(X))$ is equivalent to $\dim A \leqq \dim X$, the assertion of the subspace theorem. These statements are not always true in arbitrary topological spaces [9, p. 264]. However we obtain a subspace theorem by replacing $C(A)$ by another ring of functions associated with A. Let $C_Z(A)$ denote the set of all real-valued functions f defined on A such that for each real number r, the sets $\{x \in A \mid f(x) \leqq r\}$ and $\{x \in A \mid f(x) \geqq r\}$ are the intersections with A of zero-sets of X. Here a zero-set of X is the set of zeros of a continuous real-valued function on X. For general information about zero-sets the reader is referred to [9]. It follows from Theorem 3.5 of [10] that $C_Z(A)$ is a uniformly closed ring and is also a lattice.

The proof of the following subspace theorem will be discussed after Theorem 10.

THEOREM 1. *If A is a subspace of an arbitrary topological space X, then $d(C_Z(A)) \leqq d(C(X))$.*