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SOME ASPECTS OF AUTOMATIC CONTINUITY

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This paper deals with problems concerning the continuity of various linear maps between Banach spaces. In the first section, it is shown that if an algebra with automatic continuity properties is mapped reasonably into another algebra, that other algebra has automatic continuity properties. In the second section, properties of ideals crucial to automatic continuity questions are investigated. The third section deals with linear maps of regular commutative semi-simple Banach algebras. The last section concerns automatic continuity questions when restrictions are placed on the range of the maps.

Let X, Y be topological vector spaces, T a linear map from X into Y. Questions of automatic continuity can, in general, be stated as follows: what structural restrictions on X and Y, and what hypotheses on T, are sufficient to ensure the continuity of T? Unless X is finite-dimensional, it is, of course, necessary to place some hypotheses on the nature of T. In general, automatic continuity theorems can be classified into two major categories: Those in which the primary restrictions are placed on the domain space X, and those in which the primary restrictions are placed on the range space Y. In this paper we shall investigate both types of theorems. All maps will be assumed to be linear.

1. Separable maps and module actions.

The following concept was introduced in [9]; we present here a slight but useful generalization.

DEFINITION 1.1. Let A, B, X, Y be normed spaces, $q: A \times B \to X$ a continuous bilinear form. Let $T: X \to Y$ be linear, and let R^+ denote the positive reals. We say T is separable (with respect to q) if there exist functions $f: A \to R^+$, $g: B \to R^+$ such that $\| T(q(a, b)) \| \leqq f(a)g(b)$ for all $a \in A$, $b \in B$.

The proof of the following basic lemma is a trivial modification of ([9], Lemma 1.1), and so will be omitted.

LEMMA 1.1. *Let A, B be Banach spaces, X, Y normed spaces, $q: A \times B \to X$ a continuous bilinear form, and $T: X \to Y$ a linear map separable with respect to q. Let $\{a_n\} \subseteqq A$, $\{b_k\} \subseteqq B$ such that $q(a_n, b_k) = 0$ if $n \neq k$. Then*

$$\sup_n \| T(q(a_n, b_n)) \| / \| a_n \| \| b_n \| < \infty .$$