PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

TTF CLASSES AND QUASI-GENERATORS

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Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_A\mathscr{M}$, the category of left A-modules. In this paper the property that the torsionfree class $\mathscr{F}$ be closed under homomorphic images is investigated. Particular attention is given to the case where the torsion class $\mathscr{T}$ is torsion-torsionfree (TTF). Applications to projective quasi-generators are given.

When $\mathscr{T}$ is a TTF class the question naturally arises as to when A_t, the $\mathscr{T}$-torsion submodule of A, is contained in a certain idempotent topologizing filter of right ideals of A. This condition is shown to be equivalent to the property that the torsionfree class $\mathscr{F}$ be closed under homomorphic images. Our results generalize results of Jans [6] and Bernhardt [2] characterizing the property that the torsion theory $(\mathscr{T}, \mathscr{F})$ is centrally splitting. Dropping the assumption that $\mathscr{T}$ is TTF, further investigation of the property that $\mathscr{F}$ is closed under homomorphic images yields information as to when $\mathscr{T}$ is TTF, generalizing a result due to Rutter [10]. Finally, our methods are applied to the TTF class $\mathscr{T} = \{{}_AX \mid P \otimes {}_AX = 0\}$ where P_A is a projective right A-module. The definition of P_A being a quasi-generator is given and characterizations are obtained.

Section 2 of this paper was taken from the author's doctoral dissertation, under the direction of Professor F. L. Sandomierski, at the University of Wisconsin. Section 1 provides a generalization of the material in §2 to arbitrary TTF classes. The author expresses his gratitude to Professor Sandomierski for his guidance and encouragement.

In this paper A will be an associative ring with unit and all modules will be unitary. The left (right) annihilator of I in X will be denoted by $l_X(I)$ $(r_X(I))$. It is easy to see that for a left A-module X and a two-sided ideal I of A, $r_X(I) \cong \mathrm{Hom}_A(A/I, X)$.

Dickson [4] has defined a torsion theory for ${}_A\mathscr{M}$ to be a pair $(\mathscr{T}, \mathscr{F})$ of classes of left A-modules satisfying

(1) $\mathscr{T} \cap \mathscr{F} = \{0\}$.

(2) $\mathscr{T}$ is closed under homomorphic images.

(3) $\mathscr{F}$ is closed under submodules.

(4) For each $X \in {}_A\mathscr{M}$ there exists a (unique) submodule $X_t \in \mathscr{T}$ such that $X/X_t \in \mathscr{F}$.

A class $\mathscr{T}(\mathscr{F})$ of left modules is called a torsion (torsionfree) class provided there is a (unique) class $\mathscr{F}(\mathscr{T})$ such that $(\mathscr{T}, \mathscr{F})$ is a torsion theory. A torsion theory $(\mathscr{T}, \mathscr{F})$ is said to be hereditary if