PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

SPECTRAL DISTRIBUTION OF THE SUM OF SELF-ADJOINT OPERATORS

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Using the techniques of noncommutative integration theory, classical results of Hermann Weyl concerning the positive eigenvalues of the sum of two self-adjoint compact operators are extended to self-adjoint operators which are measurable with respect to a gage space. Let (H, A, m) be a gage space and let K and L be self-adjoint operators which are measurable with respect to (H, A, m). Let $P_K[\lambda, \infty)$ be the spectral projection of K for the interval $[\lambda, \infty)$ and let $\Lambda_K(x) = \sup\{\lambda \mid m(P_K[\lambda, \infty)) \geqq x\}$. Then $\Lambda_{K+L}(x + r) \leqq \Lambda_K(x) + \Lambda_L(r)$. If $K \leqq L$, then $\Lambda_K(x) \leqq \Lambda_L(x)$. If L is bounded, then $\Lambda_{LKL}(x) \leqq \|L\|^2 \Lambda_K(x)$ for $x \leqq m(P_K[0, \infty))$. If $q = m($support $(L))$ and $q < \infty$, then $\Lambda_K(x + q) \leqq \Lambda_{K+L}(x)$; if $\mu = \Lambda_{|K|}(q)$, then $\|K + L\|_p \geqq \|KP_K(-\mu, \mu)\|_p$ for $1 \leqq p \leqq \infty$.

1. Notation. We specifically work in the context of a gage space. [See 5 for definitions and notation.] We will always require that an operator be measurable [5, Definition 2.1]. This is a technical consideration which is necessary to avoid the pathologies which can occur with unbounded operators. Any one of the following conditions implies that a self-adjoint operator T is measurable with respect to the gage space (H, A, m):

1. $T \in A$.
2. $T\eta A$ and m is a finite gage. ($T\eta A$ means that $UT = TU$ for every unitary operator U in the commutant of A.)
3. $T\eta A$ and $m($support $(T)) < \infty$, where support (T) is the orthocomplement of the nullspace of T.
4. $T\eta A$ and A is abelian.

If P is a projection operator, P will be identified with the range of P. If T is an operator, $R(T)$ denotes the range of T and $\bar{R}(T)$ denotes the closure of $R(T)$. If T is self-adjoint, note that support $(T) = \bar{R}(T)$.

(H, A, m) is a gage space. If S and T are self-adjoint operators which are measurable with respect to (H, A, m), then $S + T(ST)$ will denote the strong sum (product) of S with T; this is the closure of the ordinary sum (product) and is self-adjoint and measurable [5, Corollary 5.2]. T has spectral decomposition $T = \int_{-\infty}^{\infty} \lambda dP_T(\lambda)$; the function $P_T(\lambda)$ is chosen to be continuous from the right. If $\mathscr{I}$ is an interval, $P_T(\mathscr{I})$ is the spectral projection of T for the interval $\mathscr{I}$. The function Λ_T is defined, for $x > 0$, by $\Lambda_T(x) = \sup\{\lambda \mid m(P_T[\lambda, \infty)) \geqq x\}$.