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THE RANGE OF A CONTRACTIVE PROJECTION ON AN L_p-SPACE

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Suppose (X, Σ, μ) is a measure space, $1 \leqq p < \infty$ and $p \neq 2$. Let $L_p = L_p(X, \Sigma, \mu)$ be the usual space of equivalence classes of Σ-measurable functions f such that $|f|^p$ is integrable. A contractive projection on L_p is a linear operator $P: L_p \to L_p$ such that $P^2 = P$ and $\|P\| \leqq 1$. In this paper we give a complete description of such contractive projections in terms of conditional expectation operators. We also show that a closed subspace M of L_p is the range of a contractive projection if and only if M is isometrically isomorphic to another L_p-space. Another sufficient condition shows, in particular, that every closed vector sublattice of an L_p-space is the range of a positive contractive projection.

Most of our results are known. The case of finite μ was treated, for $p = 1$, by Douglas [2] and for $1 < p < \infty$ by Ando [1] who showed how to reduce this case to that of $p = 1$. These authors obtained our necessary and sufficient condition. Grothendieck [4] considered $p = 1$ and general μ and showed that the range of a contractive projection on L_1 is isometrically isomorphic to another L_1-space. Wulbert [11] showed that a positive contractive projection on L_1 which is also L_∞ contractive is a conditional expectation, and pointed out that his proofs applied for $p > 1$. Tzafriri [10] showed that for general μ the range of a contractive projection on L_p is isometrically isomorphic to another L_p-space. In [5] we gave an outline, based on Tzafriri's, of another proof of this fact.

We obtain complete generalizations of the Douglas-Ando results to the case of an arbitrary measure μ. We have chosen to give our proofs in detail. It seems easier not to reduce the case $p > 1$ to the case $p = 1$. The proofs for $p > 1$ often use duality arguments which are just not available for $p = 1$. By giving such proofs, generalizations to reflexive Banach function spaces may be possible. Some such generalizations have been tried by Rao [8] but his reduction from arbitrary norms to the L_1 case is faulty and his Theorem 2.7 is false in general (see Remark 4.4). Duplissey [3] considers Banach function spaces but requires $\|Pf\|_\infty \leqq \|f\|_\infty$ as well as P contractive. We also avoid reducing to the case of finite measures. This device turns out to be unnecessary, and needlessly complicated.

We have deliberately omitted the cases $0 < p < 1$, except in the appendix, and the case $p = 2$. A contractive projection on Hilbert