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REGULARITY AND QUOTIENTS IN RINGS WITH INVOLUTION

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Let R be a ring with involution. There exists a unique maximal nilpotent *-ideal N of R such that R/N with the induced involution satisfies the property that any regular element of $\bar{S}$, the subring generated by the symmetric elements of R/N is regular in R/N. When $N = 0$ we say that R satisfies the regularity condition. Assuming this condition, $(Q(R)) = R$ if and only if $Q(\bar{S}) = \bar{S}$. The existence of $Q(\bar{S})$ implies the existence of $Q(R)$, and the converse is shown in some special cases. If either S is commutative or R is a semi-prime Goldie ring, then the relation between $Q(R)$ and $Q(\bar{S})$ is explicitly described.

Recent work on rings with involution [8] has investigated the question of when a symmetric element which is regular with respect to all other symmetrics, is regular in the ring. Our first goal here is to examine the related situation concerning the subring generated by the symmetric elements. That is, when must an element, regular in this subring, be regular in the whole ring? We show that semi-primeness is a sufficient condition on the ring, and define a nilpotent ideal N of R so that R/N possesses this regularity property. In a slightly different direction, it has been shown [7] that for a semi-prime ring, the subring generated by the symmetric elements is a Goldie ring exactly when the whole ring is a Goldie ring. This result implies that each of these rings has a semi-simple Artinian classical ring of quotients when the other does. We determine here the relation between these quotient rings, and further, investigate more general conditions under which the existence of a quotient ring for one of these rings implies the existence of a quotient ring for the other. As one might expect, this latter problem is related to the one above concerning the regularity of elements in the subring generated by the symmetric elements.

Henceforth, R will denote a ring with involution, *; $S = S(R) = \{x \in R \mid x^* = x\}$, the set of symmetric elements of R; and $\bar{S} = \bar{S}(R)$, the subring generated by S. An important fact about S is that it is a Lie ideal of R (see [4] or [6]). Thus $xt - tx \in \bar{S}$ for every $x \in R$ and $t \in \bar{S}$. An ideal I of R is called a *-ideal if $I^* = I$. Before our first result, we recall the following

DEFINITION. If A is a nonempty subset of R, then $\ell(A) = \ell_R(A) =$