PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No 1, 1975

ISOMETRIES OF THE DISK ALGEBRA

J. N. McDonald

In this paper we are concerned with the problem, posed by R. R. Phelps, of describing the into isometries of the disk algebra. We show that, in a certain sense, every isometry can be approximated by convex combinations of isometries of the form $f \to k(f \circ \phi)$. We also give some sufficient conditions for an isometry to be of the form $f \to k(f \circ \phi)$.

Let D and Γ denote, respectively, the open unit disk and the unit circle. The disk algebra, i.e., the algebra of all complex valued functions which are continuous on $D \cup \Gamma$ and analytic on D, will be denoted by A. It will be assumed that A is equipped with the sup-norm.

Operators of the form

$$Tf = k(f \circ \phi) \tag{1}$$

are isometries of A: if $k \in A$, if $\|k\| = 1$, and if $\phi: D \cup \Gamma \to D \cup \Gamma$ is analytic on D, continuous on $D \cup \Gamma \sim k^{-1}(0)$, and satisfies $\phi(k^{-1}(\Gamma)) \supset \Gamma$. In fact, if T is a surjective linear isometry of A, then it must be of the form (1) with k being a constant, and ϕ being a Mobius transformation. (See [**3**, pp. 142–148].) Rochberg [**8**] has shown that if T is an isometry such that $T1 = 1$, and $T(A)$ is a sub-algebra of A, then T is of the form (1) with $k \equiv 1$.

Note that any bounded linear operator $T: A \to A$ which satisfies (1) also satisfies.

$$T1T(fg) = TfTg \tag{2}$$

for all f and g in A. Moreover, we have the following.

Proposition 1.1. *A bounded linear operator $T: A \to A$ satisfies (2) for all $f, g \in A$ iff it is of the form* (1).

Proof. It is only necessary to show that, if T satisfies (2) for all $f, g \in A$, then it satisfies (1).

Suppose that w is a point of D where $T1$ is not 0. Consider the linear functional defined on A by

$$L_w(f) = (T1(w))^{-1}Tf(w).$$

By (2), L_w is a multiplicative. Hence, there is a v in $D \cup \Gamma$ such that