PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 1, 1976

CO-RADICAL EXTENSION OF *PI* RINGS

MAURICE CHACRON

Throughout this paper R will denote any associative ring (without necessarily 1) with a fixed subring A such that for each element x of R, there is a polynomial $g_x(t)$ (depending on x) having integral coefficients so that the element $x - x^2 \cdot g(x)$ must be in A, say, R is a *co-radical extension of the ring A*, or R *is co-radical over A*. In this paper it is shown that if A is *PI* (ring with polynomial identity) then so must be R.

Prime examples of co-radical extensions are the rings R which are co-radical over their centers $Z = Z(R)$ studied by I. N. Herstein, and the algebras R over finite fields where $A = A(R)$ is the subring generated by both the nilpotent and transcendental elements.

Essential to the paper will be both the techniques used by B. Felzenswalb, and by Herstein and L. Rowen in the study of the radical situation, that is, for each $x \in R$, $x^{n(x)} \in A$, and a recent commutativity result asserting that for any ring R the centralizer of the subring A (no *PI* assumption) must be precisely the center $Z = Z(R)$ of R.

Conventions. The center of the ring R is denoted by $Z = Z(R)$. The centralizer of the subring A in the ring R is denoted by $C_R(A)$ ($= \{a, a \in R, xa = ax, \text{ all } xeR\}$). All polynomial $g_x(t)$ considered here are polynomials with integer coefficients.

LEMMA 1. *All nilpotent elements of the ring R must be in A* (no assumption on char (R)).

Proof. Given any $x \in R$ and any $k \geqq 1$ we claim that we can find a polynomial with integer coefficients, $g_{k,x}(t)$, so that $x - x^{2^k} \cdot g_{k,x}(x) \in A$. If $k = 1$, the assertion is just our basic assumption. If true for k, then the assertion is true for $k+1$. In fact let $x_k = x^{2k} \cdot g_{k,x}(x)$. We can find $g_1(t)$ so that $x_k - (x_k)^2 \cdot g_1(x_k) \in A$. Combining these relations we obtain $x - x^{2^{k+1}} \cdot g_{k+1,x}(x) \in A$, where $g_{k+1,x}(t) = g^2_{k,x}(t) \cdot g_1(t^{2^k} \cdot g_{k,x}(t))$. It is now evident that if x is nilpotent, then $x \in A$.

LEMMA 2. *Let $a, b, x, y \in R$ with $xy = 0$ and $aAb = 0$. Then $ayRxb = 0$.*

Proof. It is clear that yRx is nil, so, by Lemma 1, must be contained in A. Thus $ayRxb \subseteq aAb = 0$, and $ayRxb = 0$ follows.