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A GENERAL RATIO ERGODIC THEOREM FOR SEMIGROUPS

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The purpose of this note is to prove a ratio ergodic theorem, which is a continuous parameter version of Chacon's general ergodic theorem.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $L_1 = L_1(X, \mathscr{F}, \mu)$ the Banach space of equivalence classes of integrable complex-valued functions on X. Let $\Gamma = \{T_t;\, t > 0\}$ be a strongly continuous semigroup of linear contractions on L_1. It then follows (cf. [**6**, §4]) that for any $f \in L_1$ there exists a scalar function $T_tf(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0,\infty)$ and $\mathscr{F}$, such that $T_tf(x)$ belongs to the equivalence class of T_tf for each $t > 0$. Moreover there exists a set $N(f)$ with $\mu(N(f)) = 0$, dependent on f but independent of t, such that if $x \notin N(f)$, then $T_tf(x)$ is integrable on every finite interval (a, b) and the integral $\int_a^b T_tf(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_tf\,dt$.

THEOREM. *Let $p_t(x)$ be a nonnegative function on $(0,\infty)\times X$, measurable with respect to the product of the Lebesgue measurable subsets of $(0,\infty)$ and $\mathscr{F}$, such that $f \in L_1$ and $|f| \leqq p_s$ for some s imply $|T_tf| \leqq p_{s+t}$ for all $t > 0$. Then for any $f \in L_1$ the limit*

$$\lim_{b\to\infty} \int_0^b T_tf(x)\,dt \Big/ \int_0^b p_t(x)\,dt$$

exists and is finite a.e. *on* $\left\{x;\, \int_0^\infty p_t(x)dt > 0\right\}$.

LEMMA. *Let T be a linear contraction on L_1 and $\{p_n;\, n \geqq 0\}$ a sequence of nonnegative measurable functions on X such that $f \in L_1$ and $|f| \leqq p_n$ for some n imply $|Tf| \leqq p_{n+1}$. If $g \in L_1$, then*

$$\lim_n p_n(x) \Big/ \sum_{i=0}^{n-1} p_i(x) = 0$$

a.e. *on*

$$\left\{x;\, \sum_{i=0}^{\infty} p_i(x) > 0 \text{ and } \lim_n \left| \sum_{i=0}^{n} T^i g(x) \Big/ \sum_{i=0}^{n} p_i(x) \right| > 0\right\}.$$