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BANACH SPACES WHICH SATISFY LINEAR IDENTITIES

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In 1935, Jordan and von Neumann proved that any Banach space which satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \text{for all elements } x \text{ and } y \tag{1}$$

must be a Hilbert space.

Subsequent authors have found norm conditions weaker than (1) which require a Banach space to be a Hilbert space. Notable examples include the results of Day, Lorch, Senechalle and Carlsson.

In this paper, we study nontrivial linear identities such as

$$\sum_{k=0}^{m} a_k \|c_k(0)x_0 + \cdots + c_k(n)x_n\|^p = 0 \text{ for all elements } x_i \tag{2}$$

on a Banach space X.

A necessary condition for (2) to hold in X is that $\|x+ty\|^p$ must be a polynomial in t for all choices of elements x and y. A sufficient condition for (2) to hold in X is that (2) must hold in the field of scalars. Specific identities are presented including a generalized parallelopiped law first observed by Koehler, and some isometric results are stated.

2. The parallelogram law revisited. In 1909 [4], Fréchet proved the following result.

LEMMA 1 (Fréchet). *If g is continuous function on $\boldsymbol{R}$ and, for all real r and s, equation* (3) *holds, then g is a polynomial with degree less than N.*

$$\sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} g(r+ks) = 0 . \tag{3}$$

Proof. It is well-known that any sequence $\{a_n\}$ satisfying $\Sigma(-1)^{N-k}\binom{N}{k}a_{k+M} = 0$ for all M is generated by a polynomial; that is, there is a polynomial P with degree less than N for which $a_n = P(n)$.

In (3), put $g(n) = a_n$, $s = 1$ and let r range over the integers. Then there is a polynomial P with $P(n) = a_n = g(n)$. Now put $g(n/2) = b_n$, $s = 1/2$ and let r range over the half-integers. There is