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THE DEGREE OF MONOTONE APPROXIMATION

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Jackson type theorems are obtained for generalized monotone approximation. Let $E_{n,k}(f)$ be the degree of approximation of f by nth degree polynomials with kth derivative nonnegative on $[-1/4, 1/4]$. Then for each $k \geqq 2$ there exists an absolute constant D_k, such that for all $f \in C[-1/4, 1/4]$ with kth difference nonnegative on $[-1/4, 1/4]$; $E_{n,k}(f) \leqq D_k \omega(f, n^{-1})$. If in addition $f' \in C[-1/4, 1/4]$ then $E_{n,k}(f) \leqq D_k n^{-1} \omega(f', n^{-1})$.

Given a function f with nonnegative kth difference on $[-1/4, 1/4]$ (equivalently any finite real interval) it is natural to ask whether Jackson type estimates hold for

$$E_{n,k}(f) = \inf_{\{p \in \Pi_n : p^{(k)}(x) \geqq 0, x \in [-1/4, 1/4]\}} \|f - p\|$$

where the norm is the uniform norm, and Π_n is the space of algebraic polynomials of degree not exceeding n. In the case $k = 1$, Lorentz and Zeller [4] and Lorentz [5] have shown that there exists a constant D_1 such that if f is increasing on $[-1/4, 1/4]$

$$E_{n,1}(f) \leqq D_1 \omega(f, n^{-1}), \qquad n = 1, 2, \cdots, \tag{1}$$

where $\omega(f, \cdot)$ denotes the modulus of continuity of f. If, in addition, $f' \in C[1/4, 14]$ then

$$E_{n,1}(f) \leqq D_1 n^{-1} \omega(f', n^{-1}), \qquad n = 1, 2, \cdots. \tag{2}$$

DeVore [2, 3] has given a much simpler proof of the $k = 1$ results. The results of this paper are obtained with similar arguments.

NOTATION. Throughout $C_1, C_2, \cdots$ denote positive constants depending on k, but not depending on f, x or $n \geqq k$. Whenever it causes no confusion, $\|\cdot\|_\beta$ denotes $\|\cdot\|_{[-\beta,\beta]}$ and $\omega(e, \cdot)$ denotes $\omega_{[-1/4, 1/4]}(e, \cdot)$.

A function with nonnegative kth difference on $[a, b]$ cannot, in general, be extended to a function with nonnegative kth difference on a larger interval. For example the piecewise linear and convex function, $f \in C[0, \sum_{n=1}^{\infty} n^{-3}]$, with slope n on the interval

$$\left[\sum_{i=1}^{n-1} i^{-3}, \sum_{i=1}^{n} i^{-3}\right],$$

cannot be extended to the right and remain convex. This motivates the construction of a preapproximation (see Lemma 1) to f, to which