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THE SIMPLEST CLOSED 3-MANIFOLDS

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Every closed orientable 3-manifold has a Heegaard diagram and corresponding group presentations. We shall show here how to give a complete analysis of all closed orientable 3-manifolds that have genus two Heegaard diagrams having a corresponding presentation one of whose relators contains no more than 4 syllables. This is equivalent to saying that one of the determining simple closed curves in the diagram crosses a "waist band" of the diagram no more than 4 times. In the appendix we have given a catalog of manifolds with two generator presentations and not more than 20 syllables.

The catalog was produced with the aid of a computer using the techniques developed in [6] and [7]. The analysis comes to a grinding halt when "generalized knot spaces" are encountered because the author has not been able to show nontriviality of the groups encountered nor has he been able to decide which of these spaces are sufficiently large. The techniques used here have been sufficient for establishing homeomorphism between any pair of orientable 3-manifolds known to be homeomorphic. The author wishes to thank H. Zieschang for helpful conversations.

1. Preliminary definitions and theorems. Let M denote a closed 3-manifold. A 2-complex K is called a spine of M if $M - B$ collapses to K, B denotes a (polyhedral) ball in M. In our discussion all spines will be assumed to have a cell decomposition with only one 0-cell (vertex). It is a simple matter to modify any spine by shrinking a maximal tree in the 1-skelton so that it has only one o-cell. It is easy to see how one obtains a group presentation from such a cell complex. The generators are in 1-1 correspondence with the 1-cells and the relators are read off from the formula by which the 2-cells attach to the 1-skelton. This group presentation will be called a *presentation of the spine.* Unfortunately the spine (and hence its presentation) does not always uniquely determine the manifold. For instance $\langle a \mid a^7 \rangle$ is a presentation of a spine of the lens spaces L_{7P} for $P = 1, 2, 3$. These spaces are not homeomorphic [11]. However for spines whose presentations have two generators and relators all of whose exponents are not ± 1 and ± 2, the spines uniquely determine the manifolds [5].

There are 2 quite common ways of building 3-manifolds-Heegaard diagrams and handle decompositions. A Heegaard diagram (H_1, H_2, h)