PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

CONTRACTION SEMIGROUPS IN LEBESGUE SPACE

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Let $(T_t: t>0)$ be a strongly continuous semigroup of linear contractions on $L_1(X, \Sigma, \mu)$, where (X, Σ, μ) is a σ-finite measure space. Without assuming the initial continuity of the semigroup it is shown that $(T_t: t>0)$ is dominated by a strongly continuous semigroup $(S_t: t>0)$ of positive linear contractions on $L_1(X, \Sigma, \mu)$, i.e., that $|T_t f| \leqq S_t |f|$ holds a.e. on X for all $f \in L_1(X, \Sigma, \mu)$ and all $t>0$. As an application, a representation of $(T_t: t>0)$ in terms of $(S_t: t>0)$ is obtained, and the question of the almost everywhere convergence of $1/b \int_0^b T_t f dt$ as $b \to +0$ is considered.

Introduction. Let (X, Σ, μ) be a σ-finite measure space and let $L_p(X) = L_p(X, \Sigma, \mu)$, $1 \leqq p \leqq \infty$, be the usual Banach spaces of real or complex functions on (X, Σ, μ). For a set $A \in \Sigma$, $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. If $f \in L_p(X)$, we define supp f to be the set of all $x \in X$ at which $f(x) \neq 0$. Relations introduced below are assumed to hold modulo sets of measure zero. A linear operator T on $L_p(X)$ is called a *contraction* if $\|T\|_p \leqq 1$, and *positive* if $f \geqq 0$ implies $Tf \geqq 0$.

Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$, i.e.,

(i) each T_t is a linear contraction on $L_1(X)$,
(ii) $T_t T_s = T_{t+s}$ for all $t, s > 0$,
(iii) for every $f \in L_1(X)$ and every $s>0$, $\lim_{t\to s} \|T_t f - T_s f\|_1 = 0$.

Under the additional hypothesis of strong-$\lim_{t\to+0} T_t = I$ (I denotes the identity operator), Kubokawa [6] proved that there exists a strongly continuous semigroup $(S_t: t > 0)$ of positive linear contractions on $L_1(X)$ such that $|T_t f| \leqq S_t |f|$ a.e. on X for all $f \in L_1(X)$ and all $t > 0$. The main purpose of this paper is to prove the same result, without assuming any additional hypothesis. We then obtain a representation of $(T_t: t > 0)$ in terms of $(S_t: t > 0)$ which is a continuous extension of Akcoglu-Brunel's representation ([1], Theorem 3.1), and a decomposition of the space X for $(T_t: t>0)$ which asserts the existence of a set $Y \in \Sigma$ such that $T_t f \in L_1(Y)$ for all $f \in L_1(X)$ and all $t > 0$ and also such that if $f \in L_1(Y)$ then $T_t f$ converges in the norm topology of $L_1(X)$ as $t \to +0$ and $1/b \int_0^b T_t f dt$ converges a.e. on X as $b \to +0$.