PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

BANACH SPACES WITH POLYNOMIAL NORMS

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A Banach space X is said to be in the class $\mathscr{P}_{2n}$ if, for all elements x and y, $||x+ty||^{2n}$ is a polynomial in real t. These spaces generalize L_{2n} and are precisely those Banach spaces in which linear identities can occur. We shall discuss further properties of $\mathscr{P}_{2n}$ spaces, often in terms of the permissible polynomials $p(t)=||x+ty||^{2n}$. For each n, the set of such polynomials forms a cone. All spaces in $\mathscr{P}_2$ are Hilbert spaces. If X is a two-dimensional real space in $\mathscr{P}_4$, then it is embeddable in L_4. This is not necessarily true for spaces with more dimensions or for $\mathscr{P}_{2n}$, $n \geqq 3$. The question of embeddability is equivalent to the classical moment problem. All spaces in $\mathscr{P}_{2n}$ are uniformly convex and uniformly smooth and thus reflexive. They obey generally weaker versions of the Hölder and Clarkson inequalities. Krivine's inequalities, shown to determine embeddability into L_p, $p \neq 2n$, fail in the even case.

1. Introduction. Throughout, we shall consider real Banach spaces, and, except where indicated, $L_{2n}(Y, \mu)$ with real-valued functions and real scalars. The phrase "X is embeddable in L_{2n}" is an abbreviation for "X is isometrically isomorphic to a subspace of $L_{2n}(Y, \mu)$ for some (Y, μ)." Although $\mathscr{P}_{2n}$ was introduced and motivated in [11], that paper and this one are largely independent.

2. Norm functions. Suppose $X = \langle x_1, \cdots, x_m \rangle$ is the real vector space spanned by the x_i's and ϕ is a real function of m real variables. Under what circumstances does $||\Sigma u_i x_i|| = \phi(u_1, \cdots, u_m)$ make $(X, ||\cdot||)$ a Banach space? For $\boldsymbol{u} = (u_1, \cdots, u_m)$, let $\phi(\boldsymbol{u}) = \phi(u_1, \cdots, u_m)$. From the standard definition of the norm, it is evident that conditions (A), (B) and (C) are necessary and sufficient. (Here, t is an arbitrary real.)

(A) $\phi(\boldsymbol{u}) \geqq 0$ and $\phi(\boldsymbol{v}) = 0$ implies $\phi(\boldsymbol{u}) \equiv \phi(\boldsymbol{u} + t\boldsymbol{v})$

(B) $\phi(t\boldsymbol{u}) = |t|\phi(\boldsymbol{u})$

(C) $\phi(\boldsymbol{u}) + \phi(\boldsymbol{v}) \geqq \phi(\boldsymbol{u} + \boldsymbol{v})$.

Condition (C) is cumbersome to verify; the following lemma simplifies matters.

LEMMA 1. *Conditions* (A), (B) *and* (C) *are equivalent to* (A), (B) *and* (D).

(D) $\psi(t) = \phi(\boldsymbol{u} + t\boldsymbol{v})$ *is a convex function in t for all $\boldsymbol{u}$ and $\boldsymbol{v}$.*

Proof. Assume (A), (B) and (C) and fix $\boldsymbol{u}$ and $\boldsymbol{v}$. Then for $0 \leqq$