PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 2, 1979

A GEOMETRIC INEQUALITY WITH APPLICATIONS TO LINEAR FORMS

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Let C_N be a cube of volume one centered at the origin in R^N and let P_K be a K-dimensional subspace of R^N. We prove that $C_N \cap P_K$ has K-dimensional volume greater than or equal to one. As an application of this inequality we obtain a precise version of Minkowski's linear forms theorem. We also state a conjecture which would allow our method to be generalized.

1. Introduction. Let $C_N = [-1/2, 1/2]^N$ be the N-dimensional cube of volume one centered at the origin in $\boldsymbol{R}^N$ and suppose that P_K is a K-dimensional linear subspace of $\boldsymbol{R}^N$. Dr. Anton Good has conjectured that the K-dimensional volume of $P_K \cap C_N$ is always greater than or equal to one. In case $K = N - 1$ this has recently been proved by Hensley [6], who also obtained upper bounds for this volume. Our purpose in this paper is to prove the conjecture for arbitrary K and to give some applications to Minkowski's theorem on linear forms. In fact we prove a more general inequality for the product of spheres of various dimensions which contains the conjecture as a special case.

We write $\bar{x}$ for the column vector $\begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ in $\boldsymbol{R}^n$ and

$$|\bar{x}| = \left(\sum_{j=1}^{n} (x_j)^2\right)^{1/2}$$

for its length. We define the sphere S_n by

$$S_n = \{\bar{x} \in \boldsymbol{R}^n : |\bar{x}| \leqq \rho_n\}$$

where $\rho_n = \pi^{-1/2}\{\Gamma(n/2 + 1)\}^{1/n}$. It follows that $\mu_n(S_n) = 1$ where μ_n is Lebesgue measure on $\boldsymbol{R}^n$. Also we let $\chi_U(\bar{x})$ denote the characteristic function of a subset U in $\boldsymbol{R}^n$.

Our first main result is contained in the following theorem.

THEOREM 1. *Suppose that $n_1, n_2, \cdots, n_J$ are positive integers, $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $\boldsymbol{R}^N$, $N = n_1 + n_2 + \cdots + n_J$, and A is a real $N \times K$ matrix,* rank$(A) = K$. *Then*

$$|\det A^T A|^{-1/2} \leqq \int_{\boldsymbol{R}^K} \chi_{Q_N}(A\bar{x})\, d\mu_K(\bar{x}) \;, \tag{1.1}$$

where A^T is the transpose of A.