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FIBER HOMOLOGY AND ORIENTABILITY OF MAPS

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In this paper we introduce a concept of *fiber homology* for an arbitrary map $f: X \to Y$ and coefficient module G. This is a graded module denoted by $H_*(f_*; G)$ which reduces to $H_*(F; G)$ when f represents an orientable fiber bundel with standard fiber F. The concept of fiber homology permits us also to define a generalized notion of orientability, and these ideas turn out to be useful in the study of submersions. Our main theorem (obtained by means of a spectral sequence) asserts that if the fibers of a submersion $f: X \to Y$ are acyclic in dimensions smaller than q, then the rank r_q of the fiber homology $H_q(f_*; G)$ is bounded above by the sum of the q and $(q+1)$-dimensional Betti numbers of X and Y, respectively. In the orientable case, the q-dimensional Betti number of an arbitrary fiber $f^{-1}(y)$ is bounded above by r_q, and therefore also by the aforementioned sum. This leads to a number of more specialized results. For example, it is shown that the fibers of an orientable submersion $f: R^{2m-1} \to S^m$ must be either acyclic or homology spheres, and moreover, the subspace of points in S^m corresponding to the spherical fibers must have the homology of a point.

1. Basic concepts. Let $f: X \to Y$ denote a continuous map between topological spaces. By a *tubular neighborhood* belonging to f we will understand a homeomorphism

$$\Phi: B \times F \approx V$$

where B is an open connected subspace of Y, F a compact space and V a subset of X, such that $f \circ \Phi$ is the projection $B \times F \to B$. Given a point $y \in B$, we will write

$$F_y = V \cap f_y$$

where f_y denotes the preimage of y under f, and given two points $y, y' \in B$, we let

$$\Phi_y^{y'}: F_y \approx F_{y'}$$

denote the homeomorphism induced by Φ. The diagram

$$\begin{array}{ccc} H_*(f_y; G) & & H_*(f_{y'}; G) \\ {\scriptstyle i}\uparrow & & {\scriptstyle i'}\uparrow \\ H_*(F_y; G) & \xrightarrow{\Phi_{y*}^{y'}} & H_*(F_{y'}; G) \end{array}$$