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HOLOMORPHY ON SPACES OF DISTRIBUTION

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If E is a locally convex space and $U \subset E$ is open, then $H(U)$ is the space of holomorphic functions on U (i.e., $H(U) = \{f: U \to C,\ f$ G-analytic and continuous). τ_0 is the topology of uniform convergence on compact subsets of U. τ_ω is the Nachbin ported topology defined by all semi-norms on $H(U)$ ported by compact subsets of U. (A semi-norm p on $H(U)$ is ported by K if whenever V is open and $K \subset V \subset U$, there exists C_V such that $p(f) \leq C_V |f|_V$ for all $f \in H(U)$.) τ_δ is the topology defined by all semi-norms p on $H(U)$ with the following property: if (U_n) is a countable increasing open cover of U, there exist $C > 0$ and U_N such that $p(f) \leqq C|f|_{U_N}$ for all $f \in H(U)$. $H_{HY}(U)$ is the space of hypoanalytic functions on U— that is $H_{HY}(U) = \{f: f$ is G-analytic and the restriction of f to any compact set $K \subset U$ is continuous\}.

If Ω is open in R^n, then $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$ are respectively the Schwartz space of test functions and the Schwartz space of distributions on Ω. We prove that $H(\mathscr{D}(\Omega)) \neq H_{HY}(\mathscr{D}(\Omega))$ and that $\tau_0 = \tau_\omega = \tau_\delta$ on $H(\mathscr{D}(\Omega))$ while $H_{HY}(\mathscr{D}'(\Omega)) = H(\mathscr{D}'(\Omega))$ but $\tau_0 \neq \tau_\omega \neq \tau_\delta$ on $H(\mathscr{D}'(\Omega))$.

1. Holomorphic and hypoanalytic functions on countable direct sums. In this section, we will prove two lemmas which are useful in the construction of holomorphic and hypoanalytic functions on countable direct sums. If $E_n \neq 0$ is a locally convex space for each n, then $E = \sum_{n=0}^{\infty} E_n$ is the countable direct sum of the E_n with the finest locally convex topology such that $E_n \to E$ is continuous for all n. $\prod_{n=0}^{\infty} E_n$ is the product of the E_n with the product topology.

LEMMA 1. *Let $E = \sum_{n=0}^{\infty} E_n = E_0 \oplus E_1 \oplus E_2 \oplus \cdots$ where each $E_n \neq 0$. For each $n > 0$, let $\psi_n \in E_n'$, $\psi_n \neq 0$. Let $(\phi_n)_{n=1}^{\infty} \subseteqq E_0'$. Then $p = \sum_{n=1}^{\infty} \phi_n \psi_n \in P_{HY}(^2E)$* (i.e., *is a hypocontinuous polynomial of degree 2 on E), and $p \in P(^2E)$ if and only if there exists an absolutely convex neighborhood V_0 of zero in E_0 such that $|\phi_n|_{V_0} < +\infty$ for each n,* (i.e., $(\phi_n)_{n=1}^{\infty} \subseteqq E_0'(V_0)$).

Proof. Since each compact subset of E is contained in $E_0 \oplus \cdots \oplus E_m$ for some m, it follows that p is a hypoanalytic polynomial of degree 2. Now if p is continuous ($p \in P(^2E)$) there exists an absolutely convex neighborhood of zero $V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots$ such that $|p|_V \leqq 1$. Now if $n \geqq 1$ is given and $y_n \in V_n$ is such that $\psi_n(y_n) \neq 0$, then $|\phi_n|_{V_0} \leqq 1/|\psi_n(y_n)|$.