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DUAL MAPS OF JORDAN HOMOMORPHISMS AND *-HOMOMORPHISMS BETWEEN C^*-ALGEBRAS

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A geometric characterization of the dual maps of Jordan homomorphisms and *-homomorphisms between C^*-algebras is given.

Introduction. In [2] the authors gave a geometric characterization of state spaces of (unital) C^*-algebras among compact convex sets. They defined the notion of an orientation of the state space, and showed that the state space as a compact convex set with orientation completely determines the C^*-algebra up to *-isomorphism. Our purpose here is to show that this correspondence is categorical by giving a geometric description of the dual maps on the state space induced by unital *-homomorphisms. Along the way we will also characterize dual maps of unital Jordan homomorphisms between C^*-algebras, and in fact in the larger category of JB-algebras: the normed Jordan algebras introduced in [3]. Finally we remark that the first result on this topic was Kadison's [6]: the dual maps of Jordan isomorphisms are precisely the affine homeomorphisms of the state spaces.

Characterization of Jordan homomorphisms. Throughout this paper A will be a C^*-algebra with state space K. (All C^*-algebras mentioned are assumed to be unital.) Assume that $A \subseteq B(H)$ is given in its universal representation, and thus its weak closure can be identified with its bidual A^{**}, and K can be identified with normal state space of A^{**} [4, §12].

We will view elements of A and A^{**} as affine functions on K. In fact, the self-adjoint parts of A and A^{**} are respectively isometrically order isomorphic to the spaces $A(K)$ and $A^b(K)$ of w^*-continuous (respectively, bounded) affine functions on K [6]. If B is also a C^*-algebra and $\phi: A \rightarrow B$ is a unital positive map then the dual map ϕ^* is an affine map from the state space K_B of B into $K = K_A$, and is weak *-continuous; $\phi \rightarrow \phi^*$ is a $1-1$ correspondence of unital positive maps and w^*-continuous affine maps. Our purpose in this section is to characterize those affine maps from K_B into K_A which correspond to Jordan homomorphisms of A into B. (In the case that the C^*-subalgebra generated by $\phi(A)$ is all of B, another characterization of the dual map has been given by Størmer [10].)

Recall that a convex subset F of K is a *face* of K if $\lambda\sigma + (1-\lambda)\tau \in F$ for $\sigma, \tau \in K$ and $\lambda \in (0, 1)$ implies σ and τ are in F. If