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ON THE ISOLATION OF ZEROES OF AN ANALYTIC FUNCTION

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The purpose of this paper is to describe extensions of the work of Errett Bishop on the location of zeroes of complex-valued analytic functions. The main result deals with the number of zeroes of an analytic function f near the boundary of a closed disc well contained in the domain of f. A particular consequence of this result is the following theorem.

Let f be analytic and not identically zero on a connected open subset U of C, K a compact set well contained in U, and $\varepsilon>0$. Then either $\inf\{|f(z)|: z \in K\}>0$ or there exist finitely many points $z_1, \cdots, z_n$ of U and an analytic function g on U such that

$$f(z)=(z-z_1)\cdots(z-z_n)g(z) \qquad (z \in U),$$

$\inf\{|g(z)|: z \in K\}>0$ **and** $d(z_k, K)<\varepsilon$ **for each** k.

The paper is written entirely within the framework of Bishop's constructive mathematics.

As Bishop has remarked [1, p. 112], the constructive development of the elementary theory of analytic functions of one complex variable presents comparatively few serious difficulties. One topic in which difficulties do arise, however, is that of location of zeroes of an analytic function. In this paper, we derive several results which apply and strengthen those obtained by Bishop [1, Ch. 5, § 5]. and which present different constructive facets of the classical theorem that the zeroes of an analytic function are isolated.

For the reader who knows little about the spirit or aims of modern constructive mathematics, we recommend Allan Calder's recent article in Scientific American [4]. The necessary technical background in constructive analysis is found in [1] and [2]; in particular, we shall assume knowledge of Chapter 5 of [1]. However, it is expedient to recall here two definitions from that chapter.

A compact subset K of an open set U in C is *well contained* in U if there exists $r>0$ such that $\{z \in C: d(z, K) \leqq r\} \subset U$; in which case we write $K \subset\subset U$. If K is a compact subset of C, then a *border* for K is a compact subset B of K such that $\bar{B}(z, d(z, B)) \subset K$ for each z in K. (We write $\bar{B}(z, r)$ for the closed disc, and $B(z, r)$ for the open disc, of center z and radius r.)

We should also note that a complex-valued function f is *not identically zero* on a set A if there exists z in A with $|f(z)|>0$;