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INTERPOLATION, CONTINUATION, AND QUADRATIC INEQUALITIES

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1. C. H. FitzGerald considered conditions for interpolating the values of an analytic function in terms of quadratic inequalities. In addition, he used them to obtain an interesting result concerning analytic continuation of functions of two variables and related them to Pick-Nevanlinna interpolation.

We shall show that these theorems follow directly from well-known principles of functional analysis. Since these principles are not limited to interpolating values of an analytic function, we shall obtain applications to interpolating functional values of both analytic and harmonic functions of several variables. In addition, we obtain analogous applications to analytic and harmonic continuation and to Pick-Nevanlinna interpolation.

2. **Abstract framework.** Let X be a reflexive topological vector space over the real or complex field and X^* its topological dual space. Assume that $p(x^*)$ is a continuous semi-norm defined on X^*. A classical theorem of Helly and Hahn (see G. Köthe [4, p. 113]) is the following. Since the proof is brief, we include it for completeness.

THEOREM 2.1 (*Helly-Hahn*). *Let $\{c_i\}_{i\in I}$ be a set of scalars and $\{x_i^*\}_{i\in I}$ a corresponding set of functionals in X^*. Then there exists an $x\in X$ such that*

$$\text{(i)}\quad x_i^*(x)=c_i \quad \textit{for all}\quad i\in I\,,\quad \textit{and}$$
$$\text{(ii)}\quad |x^*(x)|\leqq p(x^*)\quad \textit{for all}\quad x^*\in X^*\,,$$

if and only if

$$\left|\sum_{k=1}^{N}\alpha_{i_k}c_{i_k}\right|\leqq p\Big(\sum_{k=1}^{N}\alpha_{i_k}x_{i_k}^*\Big) \tag{1}$$

for every finite collection of scalars $\alpha_{i_1},\cdots,\alpha_{i_N}$.

Proof. The necessity is trivial. For the sufficiency, let A be the linear span of $\{x_i^*\}_{i\in I}$. Define a linear functional x on A by setting $x(\sum_{k=1}^N\alpha_{i_k}x_{i_k}^*)=\sum_{k=1}^N\alpha_{i_k}c_{i_k}$. Then x is well defined since if $\sum\alpha_{i_k}x_{i_k}^*=\sum\beta_{i_\kappa}x_{i_\kappa}^*$, then (1) implies

$$|\sum\alpha_{i_k}c_{i_k}-\sum\beta_{i_\kappa}c_{i_\kappa}|\leqq p(\sum\alpha_{i_k}x_{i_k}^*-\sum\beta_{i_\kappa}x_{i_\kappa}^*)=p(0)=0\,.$$