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FIXED POINT SETS OF PRODUCTS AND CONES

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A space X is said to have the complete invariance property (CIP) if every nonempty closed subset of X is the fixed point set of some self-map of X. Examples are given to show that for the class of locally connected continua, the operations of taking products, cones, and strong deformation retractions need not preserve CIP. In fact, it is shown that the operations of taking products and cones do not preserve CIP for LC^∞ continua.

1. Introduction. A subset K of a space X is called a *fixed point set* of X if there is a continuous function $f: X \to X$ such that $f(x) = x$ iff $x \in K$. In [8, p. 553] L. E. Ward, Jr. defines a space X to have the *complete invariance property* (CIP) if every nonempty closed subset of X is a fixed point set of X. Examples 4.3, 5.1 and 6.1 in [4] are examples of non-locally connected continua which show that CIP is not preserved by the operations of taking strong deformation retractions, products and cones. The purpose of this paper is to construct locally connected examples showing that CIP is not preserved by the above operations and, in fact, provide examples which answer Questions 5.2, 6.3 and 6.4 in [4]. In particular, we show that if X is a locally connected continuum possessing CIP, then $X \times I$ and Cone (X) need not have CIP. Moreover, it is shown that it is possible for X to be either a 1-dimensional continuum or an LC^∞ continuum.

2. Notation and terminology. The terminology used in this paper may be found in [1]. In particular, Hilbert space E^ω with metric ρ and Euclidean n-dimensional space E^n are as defined in [1, pp. 10–11]. For $k = 1, 2, 3, \cdots$, let a_k denote the point in E^ω given by the formula $a_k = (1/k, 0, 0, 0, \cdots)$, and let a_0 denote the origin of E^ω. Let S_k^n denote the n-dimensional sphere in E^ω consisting of all the points $x = (x_1, x_2, x_3, \cdots)$ such that $\rho(x, a_k) = 1/k$ and such that $x_i = 0$ for $i > n + 1$. Then we define

$$A^n = \bigcup_{k=1}^{\infty} S_k^n,\ A_m^\infty = \bigcup_{k=m}^{\infty} S_k^k\ .$$

A point p in a space X is said to be *homotopically stable* if for every deformation $H: X \times I \to X$, $H(p, t) = p$ for all $t \in I$. For instance, a_0 is a homotopically stable point in each of the spaces