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THE LEFSCHETZ NUMBER AND BORSUK-ULAM THEOREMS

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Let M be a manifold, with or without boundary, which is dominated by a finite complex. Let G be a finite group which acts faithfully and freely on M. Let $f: M \to M$ be a G-map. Let Λ_f denote the Lefschetz number of f and let $o(G)$ denote the order of G. The main result states, under the conditions above, that $o(G)$ divides Λ_f. Even in the case of compact M this result was not widely known. We use Wall's finiteness obstruction theory to extend the result from compact M to finitely dominated M.

The remainder of the paper is devoted to various easy applications of the result. In Theorem 5 we assume that $\pi_i(M)$ is finitely generated for all $i > 1$. Then we show that if $\pi_1(M)$ has torsion, $\pi_*(M)$ cannot be only torsion.

In Theorem 6, we have a connected Lie group L acting on M and f is an L-map. We show that the orbit map $\omega: L \to M$ induces the trivial homomorphism on fundamental groups if $\Lambda_f \neq 0$. This implies that the action of L on M can be lifted to any regular covering space.

We show that any linear transformation $T: R^n \to R^n$ which commutes with the based free action of a finite group G of order greater than 2 must have a non-negative determinant (Theorem 8).

Then we come to the Borsuk-Ulam type results. We consider maps $f: (\boldsymbol{C}^{n+1} - 0) \to \boldsymbol{C}^n$. A primitive k-root of unity ξ gives rise to a free $\boldsymbol{Z}_k$-action on $\boldsymbol{C}^n$. We show that the equation $\sum_{i=0}^{k-1} \bar{\xi}^i f(\xi^i x) = 0$ always has a solution $x \in \boldsymbol{C}^{n+1} - 0$. This result gives various conditions on the degeneracy of the images of the orbit of the $\boldsymbol{Z}_k$ action in $\boldsymbol{C}^n$. In particular, we show that if $f: S^n \to R^r$ and if $n \geqq r(p-1)$, then some orbit of the $\boldsymbol{Z}_p$-action must be mapped into a point. The proof uses the equation above and Vandermonde determinants.

2. Free actions and the Lefschetz number. A manifold M (or space) is *dominated* by a finite complex K if there exists maps $f: M \to K$ and $g: K \to M$ such that $g \cdot f$ is homotopic to the identity of M. We will need various facts about finitely dominated spaces in order to prove the result that $o(G)$ divides Λ_f for noncompact M. It is easily shown that this is true for compact M. We use the theory of C.T.C. Wall, [8], to extend to the noncompact case.

LEMMA 1. *Let M be a finitely dominated manifold. The orbit*