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SEMIFREE FINITE GROUP ACTIONS ON HOMOTOPY SPHERES

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Suppose G is a finite group acting semifreely (i.e., free off the fixed set) on a finite CW complex X in the homotopy type of S^n. When X^G is also homotopy equivalent to S^n (as e.g., in $S^n \times D^K$) necessary and sufficent conditions are given to determine the degree of the inclusion $X^G \to X$. It follows that for instance, if G is the group of quaternions (nonabelian of order 8), the only attainable degrees are those $\pm 1 \pmod 8$.

0. Introduction. Suppose that G is a finite group acting cellularly and semifreely (i.e., free off the fixed set) on a finite CW-complex X, in the homotopy type of S^n. Assume that X^G is also homotopy equivalent to S^n. The degree q of the inclusion $X^G \hookrightarrow X$ is defined and by Smith Theory is relatively prime to $|G|$ (see [1; Chap. III]). We give necessary and sufficient conditions for the converse to be true. First recall that when q is relatively prime to $|G|$, the trivial $\boldsymbol{Z}\,G$-module $\boldsymbol{Z}_q$ has projective dimension one and well-defines an element, $[\boldsymbol{Z}_q]$, in $\tilde{K}_0(\boldsymbol{Z}G)$, the reduced projective class group.

THEOREM. *Let G be a finite group acting cellularly and semifreely on a finite CW complex X. Assume that both X and X^G are in the homotopy type of S^n. Let q be the degree of $X^G \hookrightarrow X$. Then $[\boldsymbol{Z}_q] = 0$ in $\tilde{K}_0(\boldsymbol{Z}G)$. Conversely, suppose q is relatively prime to $|G|$ and $[\boldsymbol{Z}_q] = 0$ in $\tilde{K}_0(\boldsymbol{Z}G)$. Then there exists an action of G, as above, in which the degree of $X^G \hookrightarrow X$ is q.*

REMARKS. (i) From [2] it follows, e.g., that the quaternions, a nonabelian group of order 8, cannot act semifreely as above with degree of the inclusion $\pm 3 \pmod 8$.

(ii) When G is cyclic and q is prime to $|G|$, $[\boldsymbol{Z}_q]$ is always zero [4; §6] and therefore it is easy to see how to construct an effective, unrestricted action of the quaternions realizing a degree 3 inclusion. In [1; page 391], Bredon gave examples of semifree (smooth) cyclic group actions realizing any (appropriate) degree. In general, of course, one cannot hope to find smooth semifree actions of this type.

(iii) The invariant introduced here is in fact the same as the invariant, $\chi(f)$, introduced in [3] (when applied to the inclusion map).