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CHARACTERS VANISHING ON ALL BUT TWO CONJUGACY CLASSES

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If the character table of a group G has a row (corresponding to an irreducible character) with precisely two nonzero entries, then G has a unique minimal normal subgroup N which is necessarily an elementary abelian p-group for some prime p. The group $G/O_p(G)$ is completely determined here. In general, there is no bound on the derived length or nilpotence class of $O_p(G)$.

1. Introduction. An old theorem of Burnside asserts that, for any group G, any irreducible character of degree greater than 1 vanishes at some element of G (for a proof of this fact, see p. 40 of [7]). The extreme case will be considered here, namely, groups G for which a character exists which vanishes on all but two conjugacy classes. Clearly no irreducible character can vanish on all but one conjugacy class (unless $|G| = 1$).

The remaining sections of this paper are devoted to determining the structure of such groups G. Specifically, §2 is devoted to some preliminary lemmas about the action of G on its unique minimal normal subgroup N. The kernel of G on N is $\mathbf{C}_G(N) = O_p(G)$ for some prime p and $G/O_p(G)$ is determined by Theorems 4.2 and 5.6. The subgroup $O_p(G)$ can be quite complicated and this, together with some examples, are discussed in §6.

2. Some preliminary results. As already mentioned in the previous section, if a group G has an irreducible character which does not vanish on only two conjugacy classes, then G has a unique minimal normal subgroup N. The first lemma of this section establishes this, in addition to some properties of the action of G on N.

LEMMA 2.1. *Let G be a group which has an irreducible character χ such that χ does not vanish on exactly two conjugacy classes of G. If $|G| > 2$ then χ is unique and is, moreover, the unique faithful irreducible character of G. In all cases, G contains a unique minimal normal subgroup N which is necessarily an elementary abelian p-group for some prime p. The character χ vanishes on $G - N$ and is nonzero on N. Finally, the action of G by conjugation on N is transitive on $N^{\#}$.*

Proof. The conclusion of the theorem is trivial if $|G| = 2$, so assume $|G| > 2$. Clearly χ does not vanish at $1 \in G$. Let $x \in G$ be chosen so that