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TOPOLOGICAL PROPERTIES OF BANACH SPACES

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Let X be a Banach space, B_X its closed unit ball. We study several topological properties of B_X with its weak topology. In particular, we consider spaces X such that $(B_X$, weak) is a Polish topological space. If X has RNP and X^* is separable, then B_x is Polish; if B_X is Polish, then X is somewhat reflexive. We also consider spaces X such that every closed subset of $(B_X$, weak) is a Baire space. This is equivalent to property (PC), studied by Bourgain and Rosenthal.

1. Introduction. Let X be a Banach space, and B_X the closed unit ball. It is well-known that B_X is metrizable in the relative weak topology of X precisely when the dual space X^* is norm separable. Moreover, $(B_X$, weak) is compact metrizable precisely when X is separable and reflexive. Here we address an intermediate question: When is $(B_X$, weak) completely metrizable?

THEOREM A. *Let X be a separable Banach space. Then the following are equivalent*: (1) $(B_X$, *weak*) *is completely metrizable*; (2) $(B_X$, *weak*) *is a Polish space*; (3) *X has property* (PC) *and is an Asplund space*; (4) $(B_X$, *weak*) *is metrizable, and every closed subset of it is a Baire space.*

Property (PC) states that for every weakly closed bounded subset A of X, the identity map $(A$, weak) $\to$ $(A$, norm) has at least one point of continuity. (The letters "PC" stand for "point of continuity".) Property (PC) is a consequence of the Radon-Nikodym property.

Examples of spaces satisfying Theorem A include: spaces with separable second dual (such as separable quasi-reflexive spaces), the predual of the James Tree space **[30]**, **[38]** (but not JT itself), and the James-Lindenstrauss spaces JL(S) modelled on separable Banach spaces S **[29]**, **[36]**. On the other hand, if X contains an isomorphic copy of c_0 or l^1, then $(B_X$, weak) is not Polish. An example of Bourgain and Delbaen **[5]** satisfies Theorem A, but has dual space isomorphic to l^1.

There is a natural non-metrizable version of the Polish property. A completely regular Hausdorff space T is said to be *Čech complete* iff it admits a complete sequence $(\mathscr{U}_n)$ of open covers. Here completeness is defined as follows: if $\mathscr{F}$ is a family of closed subsets of T such that $\mathscr{F}$ has