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ON RADICALS AND PRODUCTS

MANFRED DUGAS AND RÜDIGER GÖBEL

An Abelian group G is called cotorsion-free if 0 is the only pure-injective subgroup contained in G. If G is a cotorsion-free Abelian group, we construct a slender, $\aleph_1$-free Abelian group A such that $\operatorname{Hom}(A, G) = 0$. This will be used to answer some questions about radicals and torsion theories of Abelian groups.

0. Introduction. In this paper we will consider torsion free abelian groups from I. Kaplansky's point of view: "In this strange part of the subject anything that can conceivably happen actually does happen", cf. [**K**, p. 81]. This statement which is supported by classical results holds in an even more spectacular sense which was not expected at this time. There are many results on torsion free abelian groups which are undecidable in ZFC, the axioms of Zermelo-Fränkel set theory including the axiom of choice. The first surprising result of this kind after years of stagnation was Shelah's solution of the famous Whitehead problem [**S1**]. In this paper Shelah also constructed for the first time arbitrarily large indecomposable abelian groups, thus improving classical results of S. Pontrjagin, R. Baer, I. Kaplansky, L. Fuchs, A. L. S. Corner and others, compare [**Fu2**, Vol. II] and [**K**]. Indecomposable abelian groups are necessarily cotorsion-free with only a few exceptions. These are the cyclic groups of prime power Z_{p^n}, the Prüfer groups $Z(p^\infty)$, the group of rational numbers $\mathbf{Q}$ and the additive group J_p of p-adic integers. A group is called cotorsion-free if and only if it contains only the trivial cotorsion subgroup 0, cf. [**GW1**]. Remember that C is cotorsion (in the sense of K. H. Harrison) if $\operatorname{Ext}_{\mathbf{Z}}(\mathbf{Q}, C) = 0$. From simple properties of cotorsion groups we conclude that a group G is cotorsion-free if and only if G is torsion-free ($Z_p \not\subseteq G$), reduced ($\mathbf{Q} \not\subseteq G$) and $J_p \not\subseteq G$ for all primes p, cf. [**GW1**]. For *countable* groups cotorsion-free is the same as reduced and torsion-free. A. L. S. Corner's celebrated theorem indicates then that each ring with a countable and cotorsion-free additive structure is the endomorphism ring of some (cotorsion-free) abelian group, cf. [**Fu1**, Vol. II]. This result was extended by the authors [**DG2**] to arbitrary rings with cotorsion-free additive groups which are then realized on arbitrarily large cotorsion-free abelian groups. Using rings without non-trivial idempotents, indecomposable groups of