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A TOPOLOGICAL BOUND ON THE NUMBER OF DISTINCT ZEROS OF AN ANALYTIC FUNCTION

ROBERT F. BROWN

An old theorem concerning the number of fixed points of a map on an annulus is used to obtain a lower bound for the number of distinct zeros of an analytic function. When the function is a polynomial, the result furnishes sufficient conditions on the coefficients so that the polynomial has at least a specific number of zeros.

Let $\mathbf{C}$ denote the complex numbers and, for real numbers r, R with $0 < r < R$, let

$$A = A_{r,R} = \{\, z \in \mathbf{C} | r \leq |z| \leq R \}.$$

It has long been known [**2**] that a (continuous) map $F\colon A \to A$ has at least $|\deg(F) - 1|$ fixed points, where $\deg(F)$ denotes the degree of F (see [**3**; page 34] for a modern proof).

We will apply this theorem to complex functions in order to obtain information about the number of zeros. There are well-known results, such as Rouché's Theorem [**1**], which count the number of zeros of complex functions. These results, however, count each zero as many times as its multiplicity. In contrast, the information we obtain is always in terms of distinct zeros.

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Let Ω be a region (open, connected subset of $\mathbf{C}$) containing the origin. Suppose $f(z)$ is a complex function analytic on Ω. By Taylor's Theorem, for each positive integer k there is a polynomial $P(z)$, of degree less than or equal to k (a Taylor polynomial of $f(z)$ at the origin), such that

$$(*) \qquad f(z) = P(z) + z^{k+1}g(z)$$

for all z in Ω, where $g(z)$ is analytic on Ω. We will refer to $(*)$ as a *Taylor decomposition* of $f(z)$. The function $Q(z)$ defined by

$$Q(z) = -z^{-k}P(z)/g(z)$$

will be called a *Taylor quotient* of $f(z)$. A fixed point of $Q(z)$ is a zero of $f(z)$.