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THE BOUNDARY BEHAVIOUR OF HARMONIC UNIVALENT MAPS

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Let D denote the open unit disc in the complex plane and $f = h+\overline{g}$ a complex-valued, harmonic, univalent and orientation preserving map in D, where h and g are analytic in D. We show that $g, h \in H^\lambda$ and $f \in h^\lambda$ for some $\lambda > 0$, where H^λ (h^λ) is the Hardy space of order λ for analytic (harmonic) functions. We also study the correspondence under f between ∂D (boundary of D) and the prime ends of $f(D)$.

1. Introduction. Let D denote the open unit disk and S_H denote the class of all complex valued, harmonic, orientation-preserving, univalent functions f in D normalized by

$$f(0) = 0 \quad \text{and} \quad f_z(0) = 1. \tag{1}$$

Each $f \in S_H$ can be expressed as

$$f = h + \overline{g}$$

where $h = z + \sum_{n=2}^{\infty} a_n z^n$ and $g = \sum_{n=1}^{\infty} b_n z^n$ are analytic in D. Clunie and Sheil-Small, [**3**], studied S_H together with some geometric subclasses of S_H. They proved, among other results, that S_H is normal with respect to the topology of uniform convergence on compact subsets of D. In this paper, we study the aspect of boundary behaviour of functions in S_H. We point out that, although some of our results are stated for $f \in S_H$, conditions (1) are not needed.

It is known that, when $f \in S_H$ is also analytic, the length of the image of the radius $[0, e^{i\theta}]$, $\int_0^1 |f'(re^{i\theta})|\,dr$, is finite for all θ except for a set of logarithmic capacity zero [**8**, p. 341]. It is also known that $f \in H^\lambda$, $0 < \lambda < \frac{1}{2}$, where H^λ is the Hardy space of analytic functions of order λ [**8**, p. 127], [**5**, p. 61]. In §2, we prove that the length of the image of the radius $[0, e^{i\theta}]$ under $f \in S_H$, $\int_0^1 |\frac{\partial f}{\partial r}(re^{i\theta})|$, is finite for almost all θ. We also prove that h, $g \in H^\lambda$ and $f \in h^\lambda$ for some λ small, where h^λ is the Hardy space of harmonic functions of order λ [**4**, p. 2]. Furthermore, as a corollary, we conclude that the radial limits $\hat{h}(e^{i\theta}) = \lim_{r\to 1^-} h(re^{i\theta})$, $\hat{g}(e^{i\theta}) = \lim_{r\to 1^-} g(re^{i\theta})$ and $\hat{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exist for almost all θ [**4**, pp. 15–17].