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HYPERBOLIC GEOMETRY IN k-CONVEX REGIONS

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A simply connected region Ω in the complex plane $\mathbb{C}$ with smooth boundary $\partial\Omega$ is called k-convex ($k > 0$) if $k(z, \partial\Omega) \geq k$ for all $z \in \Omega$, where $k(z, \partial\Omega)$ denotes the euclidean curvature of $\partial\Omega$ at the point z. A different definition is used when $\partial\Omega$ is not smooth. We present a study of the hyperbolic geometry of k-convex regions. In particular, we obtain sharp lower bounds for the density λ_Ω of the hyperbolic metric and sharp information about the euclidean curvature and center of curvature for a hyperbolic geodesic in a k-convex region. We give applications of these geometric results to the family $K(k, \alpha)$ of all conformal mappings f of the unit disk $\mathbb{D}$ onto a k-convex region and normalized by $f(0) = 0$ and $f'(0) = \alpha > 0$. These include precise distortion and covering theorems (the Bloch-Landau constant and the Koebe set) for the family $K(k, \alpha)$.

1. Introduction. We study the hyperbolic geometry of certain types of convex regions called k-convex regions. It should be emphasized that our approach is geometric rather than analytic. Our work is a continuation of that of Minda ([**11**], [**12**], [**13**], [**14**]). The paper [**11**] deals with the hyperbolic geometry of euclidean convex regions, while [**12**] treats the hyperbolic geometry of spherically convex regions. A reflection principle for the hyperbolic metric was established in [**13**]; this reflection principle leads to a criterion for hyperbolic convexity that was employed in [**14**] to give a more penetrating analysis of certain aspects of the hyperbolic geometry of both euclidean and spherically convex regions.

Roughly speaking, a region Ω in the complex plane $\mathbb{C}$ is k-convex ($k > 0$), provided $k(z, \partial\Omega) \geq k$ for all $z \in \partial\Omega$. Here $k(z, \partial\Omega)$ denotes the euclidean curvature of $\partial\Omega$ at the point z. Of course, this only makes sense if $\partial\Omega$ is a closed Jordan curve of class C^2. This condition is actually sufficient for a region to be k-convex, but not necessary. Precisely, a region Ω is (euclidean) k-convex if $|a - b| < 2/k$ for any pair of distinct points $a, b \in \Omega$ and the intersection of the two closed disks of radii $1/k$ that have a and b on their boundary lies in Ω. For example, an open disk of radius $1/k$ is k-convex as is the intersection of finitely many such disks. Thus, for a region to be k-convex it must possess a certain degree of roundness.