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ON GEOMETRIC PROPERTIES OF HARMONIC Lip_1-CAPACITY

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We shall study geometric properties of the harmonic Lip_1-capacity $\kappa'_n(E)$, $E \subset \mathbf{R}^n$. It is related to functions which are harmonic outside E and locally Lipschitzian everywhere. We shall show that $\kappa'_{n+1}(E \times I)$ is comparable to $\kappa'_n(E)$ for $E \subset \mathbf{R}^n$ and for intervals $I \subset \mathbf{R}$. We shall also show that if E lies on a Lipschitz graph, then $\kappa'_n(E)$ is comparable to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}(E)$. Finally we give some general criteria to guarantee that $\kappa'_n(E) = 0$ although $\mathcal{H}^{n-1}(E) > 0$.

1. Introduction

We shall investigate some geometric properties of the harmonic Lipschitz and C^1 capacities κ'_n and κ_n in $\mathbf{R}^n$ which were introduced in [**P**]. For the definitions see Section 2. The compact null-sets of these capacities are exactly the removable sets for the corresponding classes of harmonic functions, see Section 2, and they appear very naturally in connection of harmonic approximation problems, cf. [**P**]. The analogs for them in theory of bounded analytic functions of the complex plane are the analytic capacity γ and the continuous analytic capacity α, see e.g. [**G2**].

In Section 3 we shall study sets $E \times I$ in $\mathbf{R}^{n+1}$ where E is a bounded set in $\mathbf{R}^n$ and I an interval in $\mathbf{R}$. We shall show that $\kappa'_{n+1}(E \times I)$ is comparable to $\kappa'_n(E)$ and $\kappa_{n+1}(E \times I)$ to $\kappa_n(E)$. This gives some information about the geometric measure-theoretic properties of the null-sets of κ'_n. First we note that, as for the analytic capacity, it is easy to see that if the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}(E)$ of E is zero, then $\kappa'_n(E) = 0$ and that if the Hausdorff dimension of E is greater than $n-1$, then $\kappa'_n(E) \geq \kappa(E) > 0$. Thus problems occur only when E has dimension $n-1$ and $\mathcal{H}^{n-1}(E) > 0$. Since the null-sets for γ are also null-sets for κ'_2, we can start from the many known examples where $\gamma(E) = 0$ and $\mathcal{H}^1(E) > 0$, see e.g. [**V**], [**G1**], [**G2**], [**M2**] and [**FX**], and take products with intervals to obtain various compact sets E in $\mathbf{R}^n$ with $\kappa'_n(E) = 0$ and $\mathcal{H}^{n-1}(E) > 0$. Earlier Uy in [**U2**] generalized the example and technique of Garnett from [**G1**] to