PACIFIC JOURNAL OF MATHEMATICS
Vol. 172, No. 2, 1996

SOBOLEV SPACES ON LIPSCHITZ CURVES

María Cristina Pereyra

We study Sobolev spaces on Lipschitz graphs Γ, by means of a square function of a geometric second difference. Given a function in the Sobolev space $W^{1,p}(\Gamma)$ we show that the geometric square function is also in $L^p(\Gamma)$. For $p = 2$ we prove a dyadic analogue of this result, and a partial converse.

1. Introduction.

The Sobolev space on the real line, $W^{1,p}(\mathbf{R})$, is the set of functions in $L^p(\mathbf{R})$ whose distributional derivatives are also functions in $L^p(\mathbf{R})$.

There are several characterizations of these spaces. In the early 80's Dorronsoro (see [**Do**]) gave a mean oscillation characterization of potential spaces, extending earlier results due to R.S. Stritchartz. In the late 80's, Semmes showed that the Sobolev spaces $W^{1,2}(M)$ have many of the properties of $W^{1,2}(\mathbf{R}^n)$ when M is a chord-arc surface (see [**Se**]). Dorronsoro and Semmes used square functions closely related to the square functions we use.

There is a characterization, due to E. Stein (see [**St1**] Ch.V) that involves the second differences of the given function. More precisely, let

$$\Delta_t f(x) = f(x+t) + f(x-t) - 2f(x),$$

and define the square function

$$Sf(x) = \left(\int_0^\infty |\Delta_t f(x)|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Then the following result is true (see [**St1**]):

Theorem A [Stein]. *For $1 < p < \infty$, $f \in W^{1,p}(\boldsymbol{R})$ if and only if f, $Sf \in L^p(\boldsymbol{R})$. Moreover $\|Sf\|_p \sim \|f'\|_p$.*

For $p = 2$ the proof of this theorem is just an application of Plancherel's theorem. In this case $\|Sf\|_2 = \|f'\|_2$.

It is important for applications (eg. boundary problems for PDE's) to obtain similar results when $\mathbf{R}$ is replaced by a curve Γ. Smooth curves can be treated reducing to the case $\Gamma = \mathbf{R}$ after a suitable change of variables.