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# ON THE FAILURE CYCLES FOR THE QUADRATIC NORMALITY OF A PROJECTIVE VARIETY

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**Let $X$ be a smooth projective surface and $L$ a very ample line bundle on $X$ which is not quadratically normal; set $r + 1 = h^0(X, L)$. Here we give numerical conditions on $X$ and $L$ which imply the existence of a finite subscheme $T$ of $X$ with length$(T) \geq 2s + 2$ and contained in a dimension $s \leq r - 2$ linear subspace of $P(H^0(X, L))$ and such that $L \mid T$ is not quadratically normal.**

## Introduction.

It is very classical the following problem (with several variations). Suppose that a curve $C \subset \mathbf{P}^r$ has some bad property, e.g. it is not projectively normal. Show the existence of a finite subscheme $S$ of $C$ contained in a smaller linear subspace such that $S$ explains the failure of $C$ to be projectively norrnal. In modern times there is the important paper [**4**]. Here we consider the corresponding problem when the scheme $C$ has $\dim(C) > 1$. We were also motivated from the notion of k-ampleness and k-very ampleness introduced in [**2**]. By definition these conditions fail for a scheme $C$ if and only if there is a zero dimensional subscheme $S$ of $C$ with a bad property. We were interested (see e.g. [**1**]) in showing that under suitable conditions there are many such subschemes. A natural question was if there is some bad positive dimensional proper subscheme $Y$ containing all of them for a natural reason (for example if it were the union of them) or if there was some bad "free" zero dimensional subscheme. Here we consider the condition of quadratic normality and give a positive answer if $\dim(C) = 2$ under suitable numerical conditions. These numerical conditions are strange, far from optimal and just come from the proof. We will state them below as Theorem 0.2. But first and most important: the proofs are essentially technical variations on an alternative proof ([**5**, §2.5]) of a theorem in [**4**]; hence the idea originates ultimately with Robert Lazarsfeld. After the present results were proven, we checked the references and found that exactly that subsection was deleted in the printed version [**6**] of [**5**]. After a while we decided to rewrite a little bit the paper, but to write it anyway.