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FROM THE L^1 NORMS OF THE COMPLEX HEAT KERNELS TO A HÖRMANDER MULTIPLIER THEOREM FOR SUB-LAPLACIANS ON NILPOTENT LIE GROUPS

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This paper aims to prove a Hörmander multiplier theorem for sub-Laplacians on nilpotent Lie groups. We investigate the holomorphic functional calculus of the sub-Laplacians, then we link the L^1 norm of the complex time heat kernels with the order of differentiability needed in the Hörmander multiplier theorem. As applications, we show that order $d/2+1$ suffices for homogeneous nilpotent groups of homogeneous dimension d, while for generalised Heisenberg groups with underlying space $\mathbf{R}^{2n+k}$ and homogeneous dimension $2n+2k$, we show that order $n+(k+5)/2$ for k odd and $n+3+k/2$ for k even is enough; this is strictly less than half of the homogeneous dimension when k is sufficiently large.

1. Introduction.

We begin with the classical Laplacian $(-\Delta)$ on the Euclidean space $\mathbf{R}^d$. The multiplier theorem of L. Hörmander [**Ho**] gives a sufficient condition on a function $m : \mathbf{R}^+ \to \mathbf{C}$ for the operator $m(-\Delta)$ to be bounded on $L^p(\mathbf{R}^d)$ whenever $1 < p < \infty$, namely, when m satisfies the condition that

$$\lambda^k \left|m^{(k)}(\lambda)\right| \le c \qquad \forall \lambda \in \mathbf{R}^+ \tag{1}$$

for $0 \le k \le s = [d/2]+1$ where c is a constant and $[d/2]$ is the integral part of $d/2$.

By using fractional differentiation, the value of s in condition (1) can be improved slightly but it is known that for $(-\Delta)$ on $\mathbf{R}^d$, the value cannot be improved beyond $s = d/2$. We call s the order of the Hörmander multiplier theorem.

A lot of work has been done to obtain results of this type for other operators. E.M. Stein [**St**] proved a general result for a large class of operators, but only when the function m is of Laplace transform type, a rather restrictive condition. This was later improved by M. Cowling [**Co**], using the transference method and interpolation. For the sub-Laplacian L on a homogeneous nilpotent Lie group G of homogeneous dimension d, the following