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ON THE EXISTENCE OF EXTREMAL METRICS

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We study the well known variational problem proposed by Calabi: Minimize the functional $\int_M s_g^2 dv_g$ among all metrics in a given Kahler class. We are able to establish the existence of the extremal when the closed Riemann surface has genus different from zero. We have also given a different proof of the result originally proved by Calabi that: On a closed Riemann surface, the extremal metric has constant scalar curvature on a closed Riemann surface, the extremal metric has constant scalar curvature, which originally is proved by Calabi.

1. Introduction.

In the early 80's, E. Calabi [**C1**, **C2**] proposed the following variational problem. Let M be a compact, connected, complex $n-$dimensional manifold without boundary and assume that M admits a Kähler metric g locally expressible in the form $ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}}$. Let us fix the deRham cohomology class Ω of the real valued, closed exterior $(1,1)$ form $\omega = \sqrt{-1}g_{\alpha\bar{\beta}}dz^\alpha \Lambda dz^{\bar{\beta}}$ associated to the metric g, and denote by $\mathcal{C}_\Omega$ the function space of all differentiable Kähler metrics g with the Kähler form $\omega \in \Omega$. On this function space, Calabi introduces the (non negative) real valued functional Φ which assigns to each g the integral

$$\Phi(g) = \int_M s_g^2 dv_g$$

where $dv_g = (\sqrt{-1})^n \det(g_{\alpha\bar{\beta}})\Lambda_{\alpha=1}^n(dz^\alpha \Lambda dz^{\bar{\alpha}})$ denotes the volume element in M associated with the e Kähler metric g, and

$$s_g = -g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}\log\det(g_{\lambda\bar{\mu}})$$

the scalar curvature.

The variational problem proposed by Calabi is that of minimizing the functional $\Phi(g)$ over all $g \in \mathcal{C}_\Omega$. The motivation for considering this is the fact that, as g varies in $\mathcal{C}_\Omega$, both the volume

$$V = V_g = \int_M dv_g$$