PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 2, 1996

UNIQUENESS FOR THE n-DIMENSIONAL HALF SPACE DIRICHLET PROBLEM

D. SIEGEL AND E. O. TALVILA

In $\mathbb{R}^n$, we prove uniqueness for the Dirichlet problem in the half space $x_n > 0$, with continuous data, under the growth condition $u = o(|x| \sec^\gamma \theta)$ as $|x| \to \infty$ ($x_n = |x| \cos\theta$, $\gamma \in \mathbb{R}$). Under the natural integral condition for convergence of the Poisson integral with Dirichlet data, the Poisson integral will satisfy this growth condition with $\gamma = n - 1$. A Phragmén-Lindelöf principle is established under this same growth condition. We also consider the Dirichlet problem with data of higher order growth, including polynomial growth. In this case, if $u = o(|x|^{N+1} \sec^\gamma \theta)$ ($\gamma \in \mathbb{R}$, $N \geq 1$), we prove solutions are unique up to the addition of a harmonic polynomial of degree N that vanishes when $x_n = 0$.

1. Introduction and notation.

We use the following notation. In $\mathbb{R}^n$ ($n \geq 2$) let Π_+ be the half space $x_n > 0$ and $\partial\Pi_+$ the hyperplane $x_n = 0$. For $x \in \mathbb{R}^n$, let $y \in \mathbb{R}^{n-1}$ be identified with the projection of x onto $\partial\Pi_+$. For $x \in \Pi_+$, write $x_n = |x| \cos\theta$ and $|y| = |x| \sin\theta$ ($0 \leq \theta < \frac{\pi}{2}$). Let B_ρ be the ball of radius ρ, centre the origin in $\mathbb{R}^n$, and $d\,S_{n-1}$ its surface element. A ball with centre $x \neq 0$ is denoted $B_\rho(x)$. The volume of the unit n-ball is $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$. When integrating over regions in $\mathbb{R}^{n-1}$ the integration variable is written y' and the angle between y' and y (for fixed y) is θ_1. Unit vectors are written with a caret, e.g., $\widehat{x} = x/|x|$, and $\widehat{e}_i$ is the unit vector along the ith coordinate axis. Finally, for $k \in \mathbb{Z}$, $\mathcal{P}_k$ is the set of (real) homogeneous harmonic polynomials of degree k and $\mathcal{Y}_k$ the set of (real) spherical harmonics of degree k (see [**3**]) with the proviso that $\mathcal{P}_k = \mathcal{Y}_k = \{0\}$ for $k < 0$. If g is a function on the unit sphere, then $\|g\|^2 = \int_{\partial B_1} |g(\widehat{x})|^2 \, dS_{n-1}$.

The half space Dirichlet problem is to find u satisfying

$$u \in C^2(\Pi_+) \cap C^\circ(\overline{\Pi}_+) \tag{1.1}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{1.2}$$

$$u = f, \quad x \in \partial\Pi_+, \tag{1.3}$$