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PATHS ON POLYHEDRA. II

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Continuing the author's earlier investigation, this paper studies the behavior of paths on (convex) polyhedra relative to the facets of the polyhedra. In Section 1, the polytopes which are polar to the cyclic polytopes are shown to admit Hamiltonian circuits, and the fact that they do leads to sharp upper bounds for the lengths of simple paths or simple circuits on polyhedra of a given dimension having a given number of facets. Section 2 is devoted to the conjecture, due jointly to Philip Wolfe and the author, that any two vertices of a polytope can be joined by a path which never returns to a facet from which it has earlier departed. This implies a well-known conjecture of Warren Hirsch, asserting that $n-d$ is an upper bound for the diameter of d-dimensional polytopes having n facets. The Wolfe-Klee conjecture is proved here for 3-dimensional polyhedra, and a stronger conjecture (dealing with polyhedral cell-complexes) is established for certain special cases.

Our notation and terminology are as in [10, 11, 12, 13].[1] In particular, a *polyhedron* is a set which is the intersection of finitely many closed halfspaces in a finite-dimensional real linear space, and a *d-polyhedron* is one which is d-dimensional. The *faces* of a polyhedron P are the empty set, P itself, and the intersections of P with the various supporting hyperplanes of P. Two faces are *incident* provided one contains the other. The 0-faces and 1-faces of P are its *vertices* and *edges*, and when P is a d-polyhedron its $(d-1)$-faces and $(d-2)$-faces are called *facets* and *subfacets* respectively. A *proper polyhedron* is one which contains no line, or, equivalently (assuming it is not empty), one which has at least one vertex. A *polytope* is a bounded polyhedron; equivalently, it is a set which is the convex hull of a finite set of points. Two vertices of a polyhedron P are *adjacent* provided they are joined by an edge of P. A *path* on P is a finite sequence $(x_0, x_1, \cdots, x_l)$ of consecutively adjacent vertices, and the integer l is the *length* of the path. The *diameter* of a polyhedron is the smallest number l such that any two vertices of the polyhedron can be joined by a path of length $\leqq l$.[2]

The present paper is part of a development of recent years in

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[1] Hirsch's conjecture is reported on p. 168 of [**5**]. See also p. 160 of [**5**] and pp. 608-610 of [**10**]. And see footnote 16 at end of paper.

[2] Diameters of polyhedra are studied in [**10**,**25**].