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THE PRODUCT FORMULA FOR SEMIGROUPS DEFINED BY FRIEDRICHS EXTENSIONS

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Suppose that A and B are linear operators which generate semigroups on a Hilbert space. Then $A + B$ may be far from being a generator. Nevertheless, a generator may sometimes be defined by adding operators corresponding to A and B, but with values in a larger Hilbert space, and then restricting the sum to the original Hilbert space. Here an explicit product formula in terms of the semigroups generated by A and B is shown to converge to a semigroup, which is that given by this sum. This result has application to the perturbation theory of partial differential equations. This is illustrated by the Feynman path integral representation of the solution of the Schrödinger equation with potential term containing singularities.

Let A and B be linear operators generating semigroups $\exp(tA)$ and $\exp(tB)$ in a Banach space. Then under suitable conditions

$$\exp(t(A+B)) = \lim_{n\to\infty}\left(\exp\left(\frac{t}{n}A\right)\exp\left(\frac{t}{n}B\right)\right)^n .$$

This is the Trotter [28] product formula. Though it is a theorem in perturbation theory, it is also related to the Feynman path integral representation of solutions of partial differential equations and provides the best mathematical realization of this idea presently known.

Feynman [10] considered the Schrödinger equation of quantum mechanics

$$i\frac{du(t)}{dt} = -\frac{1}{2m}\Delta u(t) + Vu(t)$$

for $u(t)$ in $L^2(\boldsymbol{R}^3)$ for each t and with initial condition $u(0) = u$. Here Δ is the Laplacian and V is a real valued function on $\boldsymbol{R}^3$ multiplying $u(t)$. If we take $A = (i/2m)\Delta$ and $B = -iV$, the Trotter formula may be applicable. We have explicitly

$$\begin{aligned}&\exp\left(\frac{it}{2m}\Delta\right)u(x)\\ &\quad = (2\pi it/m)^{-3/2}\int_{\boldsymbol{R}^3}\exp\left[i\frac{1}{2}m(|x-y|^2/t)\right]u(y)dy .\end{aligned}$$

Thus, as Nelson [24] observed, the formula representing the solution is