PACIFIC JOURNAL OF MATHEMATICS
Vol. 115, No. 2, 1984

RETRACTION METHODS IN NIELSEN FIXED POINT THEORY

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Let X be a topological space, A a subset of X, and $F\colon X \to X$ a map. Suppose there exists a retraction $\rho\colon W \to A$ where $A \cup F(A) \subseteq W \subseteq X$; then define $f\colon A \to A$ by $f(a) = \rho F(a)$. The map f is called a retract of F. If all the fixed points of f are fixed points of F, we say that F is retractible onto A (with respect to ρ). Then, if A is a compact ANR, the Nielsen number $N(f)$ of f is a lower bound for the number of fixed points of F, or of any map $G\colon X \to X$ retractible onto A with retract homotopic to f. Many classes of examples of retractible maps can be found, even if X is required to be a euclidean space. If F is retractible onto a compact ANR with respect to a deformation retraction of X onto A, then we say that F is deformation retractible (dr) and we define a number $D(F)$ which we prove to have the property: if $G\colon X \to X$ is a dr map homotopic to F, then G has at least $D(F)$ fixed points. If X is an ANR and F is a compact map, then $D(F)$ is the Nielsen number of F. We find conditions, for any map $F\colon X \to X$ retractible onto A, so that there exists $G\colon X \to X$ retractible onto A and with retract homotopic to f such that G has exactly $N(f)$ fixed points. Furthermore, if F is dr, the hypotheses yield a dr map G homotopic to F and with exactly $D(F)$ fixed points. These last results are based on a technique, of independent interest, for extending a map $g\colon A \to A$, on a finite subpolyhedron of a locally finite polyhedron X, to a map $G\colon X \to X$ in such a way that G has no fixed points on $X - A$.

1. The Poincaré-Bohl theorem. Retraction-type results are among the oldest in fixed point theory. In 1904, Bohl [**1**] proved

THEOREM 1.1. *Let $C = \{x = (x_1,\ldots,x_n) \in \mathbf{R}^n \mid |x_i| \le a_i\}$ for some positive numbers $a_1,\ldots,a_n$. If $g\colon C \to \mathbf{R}^n - 0$ is a map, then there exists a point x on the boundary of C such that $g(x) = \alpha x$, for some $\alpha < 0$.*

In 1910, Hadamard [**13**] observed that Bohl's result was equivalent to an earlier theorem of Poincaré [**28**] and therefore called it the Poincaré-Bohl Theorem. It will be convenient to restate the theorem in the form:

THEOREM 1.2. (*Poincaré-Bohl Theorem.*) *Let $F\colon \mathbf{R}^n \to \mathbf{R}^n$ be a map and suppose there exists $R > 0$ such that*

$$(*)\qquad \|x\| = R \text{ implies } F(x) \neq \lambda x, \qquad \text{for all } \lambda > 1.$$

Then $F(x) = x$ for some x with $\|x\| \le R$.