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Note on the dimension of modules and algebras

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Let Λ be a ring with unit element. The left dimension (notation: l. dim ${}_{\Lambda}A$), the left injective dimension (l inj. dim. ${}_{\Lambda}A$) and the left weak dimension (w. l. dim ${}_{\Lambda}A$) for left Λ-modules and the left global dimension (l. gl. dim Λ) and the global weak dimension (w. gl. dim Λ) of Λ are those defined in [3].

Len Λ and Γ be rings and ψ a ring homomorphism of Λ to Γ. Then each left Γ-module A may be regarded as a left Λ-module, by setting, for $\lambda \in \Lambda$ $a \in A$

$$\lambda \cdot a = \psi\lambda \cdot a$$

If Γ is Λ-projective in this sence, the following inequalities are shown in [3]; l. dim ${}_{\Lambda}A \leqq$ l. dim ${}_{\Gamma}A$, w. 1. dim ${}_{\Lambda}A \leqq$ w. 1. dim ${}_{\Gamma}A$ and 1. inj. dim ${}_{\Lambda}A \leqq$ l. inj. dim ${}_{\Gamma}A$ for left Γ-madules A.

M. Auslander [1] has shown that l. gl. dim $\Lambda =$ sup l. dim $\Lambda/\mathfrak{l}$ where $\mathfrak{l}$ ranges over all left ideals of Λ and obtained some relations among l. gi. dim Λ_1, l. gl. dim Λ_2 and l. gl. dim $\Lambda_1 \otimes \Lambda_2$ in the special cases where Λ_1 and Λ_2 are algebras over a field K.

If $\mathfrak{A}$ is a two-sided ideal in Λ, there is in general very little relation between l. gl. dim Λ and l. gl. dim $(\Lambda/\mathfrak{A})$; it was however proved in Elenberg-Nagao-Nakayama [6] that if l. gl. dim $\Lambda \leqq 1$ and Λ is semi-primary, then gl. dim $(\Lambda/\mathfrak{A}) < \infty$.

Now, we show in section 1 of the present note that for each left Λ-module A we have l. dim ${}_{\Lambda}A =$ l. dim ${}_{\Lambda_n}A^n$, w. l. dim ${}_{\Lambda}A =$ w. l. dim ${}_{\Lambda_n}A^n$ and l. inj. dim ${}_{\Lambda}A$ $=$ l. inj. dim ${}_{\Lambda_n}A^n$ and conversely, for each left Λ_n-module A, l. dim ${}_{\Lambda}A =$ l. dim ${}_{\Lambda_n}A$ and so on, where Λ_n is the total matrix ring of order n over Λ. Hence, as the special case of $\Lambda_1 \otimes \Lambda_2$ we obtain l. gl. dim $\Lambda =$ l. gl. dim Λ_n and w. gl. dim Λ $=$ w. gl. dim Λ_n for any ring Λ and further if Λ is an algebra over a commutative ring K, we obtain dim $\Lambda =$ dim Λ_n.

In section 2 we show that the analogous theorem to Auslander's is valid for w. gl. dim Λ and some characterization of ring Λ with w. gl. dim $\Lambda \leqq n$ or l. gl. dim $\Lambda \leqq n$ $(n \geqq 1)$. In section 3, we assume that ψ is a ring homomorphism of Λ to Γ and 1. dim ${}_{\Lambda}\Gamma = 0$ or r. dim ${}_{\Lambda}\Gamma = 0$, then we obtain some relations between the dimensions of Λ and Γ, regarding Γ-modules A as Λ-modules. In particular, if two sided ideal $\mathfrak{A}$ is equal to Λe or $e\Lambda$ $(e=e^2)$, we obtain l. gl. dim $\Lambda \geqq$ l. gl. dim $(\Lambda/\mathfrak{A})$ and w. gl. dim $\Lambda \geqq$ w. gl. dim $(\Lambda/\mathfrak{A})$.